Adding Counting Quantifiers to Graph Patterns

Wenfei Fan[1,2] Yinghui Wu[3] Jingbo Xu[1,2]
[1]University of Edinburgh [2]Beihang University [3]Washington State University
{wenfei@inf, jingbo.xu@}.ed.ac.uk, yinghui@eecs.wsu.edu

ABSTRACT

This paper proposes quantified graph patterns (QGPs), an extension of graph patterns by supporting simple counting quantifiers on edges. We show that QGPs naturally express universal and existential quantification, numeric and ratio aggregates, as well as negation. Better still, the increased expressivity does not come with a much higher price. We show that quantified matching, *i.e.,* graph pattern matching with QGPs, remains NP-complete in the absence of negation, and is DP-complete for general QGPs. We show how quantified matching can be conducted by incorporating quantifier checking into conventional subgraph isomorphism methods. We also develop parallel scalable algorithms for quantified matching. As an application of QGPs, we introduce quantified graph association rules defined with QGPs, to identify potential customers in social media marketing. Using real-life and synthetic graphs, we experimentally verify the effectiveness of QGPs and the scalability of our algorithms.

CCS Concepts

•**Theory of computation** → *Pattern matching;*

Keywords

Quantified graph patterns; graph association rules

1. INTRODUCTION

Given a graph pattern $Q(x_o)$ and a graph G, *graph pattern matching* is to find $Q(x_o, G)$, the set of matches of x_o in subgraphs of G that are isomorphic to Q. Here "query focus" x_o is a designated node of Q denoting search intent [9]. Traditionally, pattern Q is modeled as a (small) graph in the same form as G. This notion of patterns is used in social group detection and transportation network analysis.

However, in applications such as social media marketing, knowledge discovery and cyber security, more expressive patterns are needed, notably ones with counting quantifiers.

Example 1: (1) Consider an association rule for specifying regularities between entities in social graphs:

- *If (a) person x_o is in a music club, and (b) among the people whom x_o follows, at least 80% of them like an album y, then the chances are that x_o will buy y.*

Its antecedent specifies conditions (a) and (b). If these two conditions hold, then we can recommend album y to x_o. This is an example of social media marketing, which is predicted to trump traditional marketing. Indeed, empirical studies suggest that "90% of customers trust peer recommendations versus 14% who trust advertising" [1], and "the peer influence from one's friends causes more than 50% increases in odds of buying products" [8].

The antecedent is specified as a *quantified graph pattern* (QGP) $Q_1(x_o)$ shown in Fig. 1, where x_o is its query focus, indicating potential customers. Here edge follow(x_o, z) carries a *counting quantifier* "$\geq 80\%$", for condition (b) above. In a social graph G, a node v_x matches x_o in Q_1, *i.e.,* $v_x \in Q_1(x_o, G)$, if (a) there exists an isomorphism h from Q_1 to a subgraph G' of G such that $h(x_o) = v_x$, *i.e.,* G' satisfies the topological constraints of Q_1, and (b) among all the people whom v_x follows, 80% of them account for matches of z in $Q_1(G)$, satisfying the counting quantifier.

The following association rules are also found useful in social media marketing, with various counting quantifiers:

- *If for all the people z whom x_o follows, z recommends Redmi 2A (cell phone), then x_o may buy a Redmi 2A.*
- *If among the people followed by x_o, (a) at least p of them recommend Redmi 2A, and (b) no one gives Redmi 2A a bad rating, then x_o may buy Redmi 2A.*

The antecedents of these rules are depicted in Fig. 1 as QGPs $Q_2(x_o)$ and $Q_3(x_o)$, respectively. Here Q_2 uses a *universal* quantification ($= 100\%$), while Q_3 carries numeric aggregate ($\geq p$) and *negation* ($= 0$). In particular, a node v_x in G matches x_o in Q_3 only if there exists *no* node v_w in G such that follow(v_x, v_w) is an edge in G and there exists an edge from v_w to Redmi 2A labeled "bad rating". That is, counting quantifier "$= 0$" on edge follow(x_o, z_2) enforces negation.

(2) Quantified graph patterns are also useful in knowledge discovery. For example, QGP $Q_4(x_o)$ of Fig. 1 is to find

- *all people who (a) are professors in the UK, (b) do not have a PhD degree, and (c) have at least p former PhD students who are professors in the UK.*

It carries negation ($= 0$) and numeric aggregate ($\geq p$). □

These counting quantifiers are not expressible in traditional graph patterns. Several questions about QGPs are open. (1) How should QGPs be defined, to balance their expressive power and complexity? (2) Can we efficiently conduct graph pattern matching with QGPs in real-life graphs,

SIGMOD'16, June 26-July 01, 2016, San Francisco, CA, USA

 ISBN 978-1-4503-3531-7/16/06…$15.00
DOI: http://dx.doi.org/10.1145/2882903.2882937

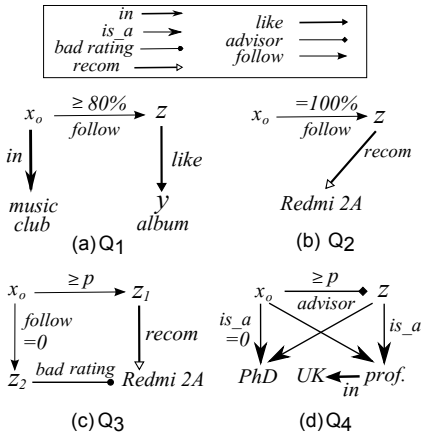


Figure 1: Quantified graph patterns

which may have trillions of nodes and edges [20]? (3) How can we make use of QGPs in emerging applications? The need for studying these is highlighted in, *e.g.,* social marketing, knowledge discovery and cyber security.

Contributions. This paper aims to answer these questions.

(1) We propose QGPs (Section 2). Using simple counting quantifiers, QGPs uniformly support numeric and ratio aggregates, universal and existential quantification, and negation. We formalize *quantified matching*, *i.e.,* graph pattern matching with QGPs, by revising the traditional semantics of pattern matching to incorporate counting quantifiers.

(2) We establish the complexity of quantified matching (Section 3). We show that despite their increased expressivity, QGPs do not make our lives much harder: quantified matching is NP-complete in the absence of negation, the same as subgraph isomorphism; and it is DP-complete otherwise.

(3) We provide a quantified matching algorithm (Section 4). The algorithm unifies conventional pattern matching and quantifier verification in a generic search process, and handles negation by novel incremental evaluation IncQMatch. As opposed to conventional incremental settings, IncQMatch acts in response to changes in patterns, not in graphs, and is *optimal* by performing only necessary verification.

(4) We develop parallel algorithms for quantified matching (Section 5). We identify a practical condition under which quantified matching is *parallel scalable*, *i.e., guaranteeing* provable reduction in sequential running time with the increase of processors. Under the condition, we develop graph partition and QGP matching algorithms, both parallel scalable, by exploring inter and intra-fragment parallelism.

(5) As an application of QGPs, we introduce quantified graph association rules (QGARs; Section 6). QGARs help us identify potential customers in social graphs, and (positive and negative) correlations in knowledge graphs. We propose support and confidence metrics for QGARs, a departure from their conventional counterparts. We also show that the (parallel) quantified matching algorithms can be readily extended to identify interesting entities with QGARs.

(6) Using real-life and synthetic graphs, we experimentally verify the effectiveness of QGPs and the scalability of our algorithms (Section 7). We find the following. (a) Quantified matching is feasible on large graphs. It takes 125 seconds on graphs of 150 millions nodes and edges by using 4 processors, and 42.3 seconds with 20 processors. (b) Our matching (resp. partition) algorithm is parallel scalable: it is on average 2.8 (resp. 3.5) and 3.2 (resp. 2.5) times faster on real-life social and knowledge graphs, respectively, when the number of processors increases from 4 to 20. (c) QGARs capture behavior patterns in social and knowledge graphs that cannot be expressed with conventional graph patterns.

We contend that QGPs and QGARs are useful in emerging applications such as social marketing and knowledge discovery. Despite the increased expressivity, they yield practical tools over large real-world graphs, which can be built upon existing (parallel) graph analytic systems.

Proofs and optimization strategies are given in Appendix.

Related work. We categorize the related work as follows.

Quantified graph querying. The need for counting in graph queries has long been recognized. SPARQLog [28] extends SPARQL with first-order logic (FO) rules, including existential and universal quantification over node variables. Rules for social recommendation are studied in [30], using support count as constraints. QGRAPH [10] annotates nodes and edges with a counting range (count 0 as negated edge) to specify the number of matches that must exist in a database. Set regular path queries (SRPQ) [31] extends regular path queries with quantification for group selection, to restrict the nodes in one set connected to the nodes of another. For social networks, SocialScope [5] and SNQL [32] are algebraic languages with numeric aggregates on node and edge sets.

The study of QGPs is to strike a balance between the expressivity and the complexity. It differs from the prior work in the following. (1) Using a uniform form of counting quantifiers, QGPs support numeric and ratio aggregates (*e.g.,* at least p friends and 80% of friends), and universal (100%) and existential quantification (≥ 1). In contrast, previous proposals do not allow at least one of these. (2) We focus on graph pattern queries, which are widely used in social media marketing and knowledge discovery; they are beyond set regular expressions [31] and rules of [30]. (3) Quantified matching with QGPs is DP-complete at worst, slightly higher than conventional matching (NP-complete) in the polynomial hierarchy [33]. In contrast, SPARQL and SPARQLog are PSPACE-hard [28], and SRPQ takes EXPTIME [31]; while the complexity bounds for QGRAPH [10], SocialScope [5] and SNQL [32] are unknown, they are either more expensive than QGPs (*e.g.,* QGRAPH is a fragment of FO(count)), or cannot express numeric and ratio quantifiers [5,32]. (4) No prior work has studied parallel scalable algorithms for its queries.

Parallel pattern matching. A number of (parallel) matching algorithms have been developed for subgraph isomorphism [22, 27, 35]. None of these addresses quantifiers. In contrast, (1) in the same general framework [27] used by these methods, our sequential quantified matching algorithms cope with quantifiers and negated edges without incurring considerable cost; and (2) our parallel scalable algorithms exploit both inter and intra-fragment parallelism for effective quantifier verification in QGP evaluation.

Various strategies have been studied for graph partition [6, 11, 23]. This work differs from the prior work in the following. (1) We propose a d-hop preserving partition scheme such that the d-hop neighbor of each node is contained in a fragment, and that all fragments have an even size, with an approximation bound. Closest to ours is the n hop-guarantee partition [22]. However, [22] provides no approxi-

mation bound to ensure both d-hop preserving and balanced fragment sizes, especially for nodes with a high degree. (2) We propose a partition algorithm that is parallel scalable, a property that is not guaranteed by the prior strategies.

Quantified association rules. Association rules [3] are traditionally defined on relations of transaction data. Over relations, quantified association rules [38] and ratio rules [25] impose value ranges or ratios (*e.g.,* the aggregated ratio of two attribute values) as constraints on *attribute values*. There has also been recent work on extending association rules to social networks [30, 36] and RDF knowledge bases, which resorts to mining conventional rules and Horn rules (as conjunctive binary predicates) [17] over tuples with extracted attributes from social graphs, instead of exploiting graph patterns. Closer to this work is [16], which defines association rules directly with patterns without quantifiers.

Our work on QGARs differs from the previous work in the following. (1) As opposed to [3, 25, 38], QGARs extend association rules from relations to graphs. They call for topological support and confidence metrics, since the conventional support metric is not anti-monotonic in graphs. (2) QGARs allow simple yet powerful counting quantifiers to be imposed on matches of graph patterns, beyond attribute values. In particular, rules of [25, 38] cannot express universal quantification and negation. When it comes to graphs, (3) the rules of [16] cannot express counting quantifiers, and limits their consequent to be a single edge, and (4) applying QGPs and QGARs becomes an intractable problem, as opposed to PTIME for conventional rules in relations.

2. QUANTIFIED GRAPH PATTERNS

We next introduce quantified graph patterns QGPs. To define QGPs, we first review conventional graph patterns.

2.1 Conventional Graph Pattern Matching

We consider labeled, directed *graphs*, defined as $G = (V, E, L)$, where (1) V is a finite set of nodes; (2) $E \subseteq V \times V$ is a set of edges, in which (v, v') denotes an edge from node v to v'; and (3) each node v in V (resp. edge e in E) carries $L(v)$ (resp. $L(e)$), indicating its label or content as commonly found in social networks and property graphs.

Two example graphs are depicted in Fig. 2.

We review two notions of subgraphs. (1) A graph $G' = (V', E', L')$ is a *subgraph of* $G = (V, E, L)$, denoted by $G' \subseteq G$, if $V' \subseteq V$, $E' \subseteq E$, and for each edge $e \in E'$ (resp. node $v \in V'$), $L'(e) = L(e)$ (resp. $L'(v) = L(v)$). (2) We say that G' is *a subgraph induced by* a set V' of nodes if $G' \subseteq G$ and E' consists of all the edges in G whose endpoints are in V'.

Patterns. A *graph pattern* is traditionally defined as a graph $Q(x_o) = (V_Q, E_Q, L_Q)$, where (1) V_Q (resp. E_Q) is a set of pattern nodes (resp. edges), (2) L_Q is a function that assigns a node label $L_Q(u)$ (resp. edge label $L_Q(e)$) to each pattern node $u \in V_Q$ (resp. edge $e \in E_Q$), and (3) x_o is a node in V_Q, referred to as the *query focus* of Q, for search intent.

Pattern matching. A *match* of pattern Q in graph G is a *bijective function* h from nodes of Q to nodes of a subgraph G' of G, such that (a) for each node $u \in V_Q$, $L_Q(u) = L(h(u))$, and (b) (u, u') is an edge in Q if and only if $(h(u), h(u'))$ is an edge in G', and $L_Q(u, u') = L(h(u), h(u'))$. From h, subgraph G' can be readily deduced.

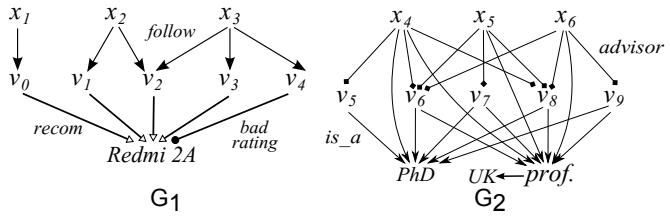


Figure 2: Graphs

We denote by $Q(G)$ *the set of matches* of Q in G, *i.e.,* the set of bijective functions h that induce a match of Q in G. *Query answer* is the set of all matches of x_o in $Q(G)$.

Given $Q(x_o)$ and G, *graph pattern matching* is to compute $Q(x_o, G)$, *i.e.,* all matches of query focus x_o in G via Q.

2.2 Quantified Graph Patterns

We next define QGPs, by extending conventional graph patterns to express quantified search conditions.

Syntax. A *quantified graph pattern* (QGP) $Q(x_o)$ is defined as (V_Q, E_Q, L_Q, f), where V_Q, E_Q, L_Q and x_o are the same as their traditional counterparts, and f is a function such that for each edge $e \in E_Q$, $f(e)$ is a predicate of

- a *positive form* $\sigma(e) \odot p\%$ for a real number $p \in (0, 100]$, or $\sigma(e) \odot p$ for a positive integer p, or
- $\sigma(e) = 0$, where e is referred to as a *negated edge*.

Here $\odot$ is either $=$ or $\geq$, and $\sigma(e)$ will be elaborated shortly. We refer to $f(e)$ as the *counting quantifier* of e, and $p\%$ and p as *ratio* and *numeric aggregate*, respectively.

Counting quantifiers express logic quantifiers as follows:

- *negation* when $f(e)$ is $\sigma(e) = 0$ (*e.g.,* Q_3 in Example 1);
- *existential quantification* if $f(e)$ is $\sigma(e) \geq 1$; and
- *universal quantifier* if $f(e)$ is $\sigma(e) = 100\%$ (*e.g.,* Q_2).

A conventional pattern Q is a special case of QGP when $f(e)$ is $\sigma(e) \geq 1$ for all edges e in Q, *i.e.,* if Q has existential quantification only. We leave out $f(e)$ if it is $\sigma(e) \geq 1$.

We call a QGP Q *positive* if it contains no negated edges (*i.e.,* edges e with $\sigma(e) = 0$), and *negative* otherwise.

Example 2: Graph patterns Q_1–Q_4 given in Example 1 are QGPs with various counting quantifiers, *e.g.,* (1) edge (x_o, z) in Q_1 has a quantifier $\sigma(x_o, z) \geq 80\%$; (2) Q_2 has a universal quantifier $\sigma(x_o, z)$=100% on edge (x_o, z), and an existential quantifier for edge $(z, \text{Redmi 2A})$; and (3) Q_3 has a negated edge (x_o, z_2) with $\sigma(x_o, z_2) = 0$. Among the QGPs, Q_1 and Q_2 are positive, while Q_3 and Q_4 are negative. □

Remark. To strike a balance between the expressive power and the complexity of pattern matching with QGPs in large-scale graphs, we assume a predefined *constant* l such that on any simple path (*i.e.,* a path that contains no cycle) in $Q(x_o)$, (a) there exist at most l quantifiers that are not existential, and (b) there exist no more than one negated edge, *i.e.,* we exclude "double negation" from QGPs.

The reason for imposing the restriction is twofold. (1) Without the restriction, quantified patterns would be able to express first-order logic (FO) on graphs. Indeed, FO sentences $P_1 X_1 \ldots P_l X_l\ \varphi$ can be encoded in such a pattern, where P_i is either $\forall$ or $\exists$, φ is a logic formula, and l is unbounded. Such patterns inherit the complexity of FO [29], in addition to #P complication. Then even the problem for deciding whether there exists a graph that matches such a pattern is beyond reach in practice. As will be seen shortly, the restriction makes QGPs discovery and evaluation feasible in

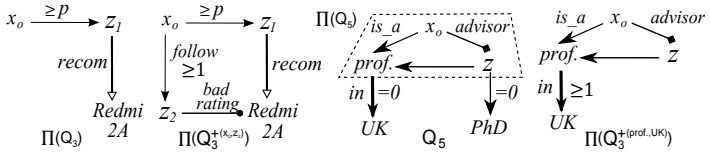


Figure 3: Negative QGPs

large-scale graphs. (2) Moreover, we find that QGPs with the restriction suffice to express quantified patterns commonly needed in real-life applications, for *small* l. Indeed, empirical study suggests that l is at most 2, and "double negation" is rare, as "99% of real-world queries are star-like" [18].

One can extend $f(e)$ to support $>$, $\neq$ and $\leq$ as $\odot$, and conjunctions of predicates. To simplify the discussion, we focus on QGPs $Q(x_o)$ in the simple form given above.

Semantics. We next give the semantics of QGPs. We consider positive QGPs first, and then QGPs with negation.

Positive QGPs. We use the following notations. Striping all quantifiers $f(e)$ off from a QGP $Q(x_o)$, we obtain a conventional pattern, referred to as the *stratified pattern* of $Q(x_o)$ and denoted by $Q_\pi(x_o)$. Consider an edge $e = (u, u')$ in $Q(x_o)$, a graph G and nodes v_x and v in G. When x_o is mapped to v_x, we define *the set of children of v via e and Q*, denoted by $M_e(v_x, v, Q)$ when G is clear from the context:

$$\{v' \mid h \in Q_\pi(G), h(x_o) = v_x, h(e) = (v, v')\},$$

i.e., the set of children of v that match u' when u is mapped to v, subject to the constraints of Q_π. Abusing the notion of isomorphic mapping, $h(e) = (v, v')$ denotes $h(u) = h(v)$, $h(u') = h(v')$, $(v, v') \in G$ and $L_Q(u, u') = L(v, v')$.

We define $M_e(v) = \{v' \mid (v, v') \in G, L(v, v') = L_Q(e)\}$, the set of the children of v connected by an e edge.

For a positive QGP $Q(x_o)$, *a match* $h_0 \in Q(G)$ satisfies the following conditions: for *each* node u in Q and *each* edge $e = (u, u')$ in Q,

- if $f(e)$ is $\sigma(e) \odot p\%$, then $\frac{|M_e(h_0(x_o), h_0(u), Q)|}{|M_e(h_0(u))|} \odot p\%$, in terms of the ratio of the number of children of v via e and Q to the total number of children of v via e; and
- if $f(e)$ is $\sigma(e) \odot p$, then $|M_e(h_0(x_o), h_0(u), Q)| \odot p$, in terms of the number of children of v via e and Q.

That is, $\sigma(e)$ is defined as ratio $\frac{|M_e(h_0(x_o), h_0(u), Q)|}{|M_e(h_0(u))|}$ or cardinality $|M_e(h_0(x_o), h_0(u), Q)|$, for $p\%$ or p, respectively. Intuitively, $\sigma(e)$ requires that at least $p\%$ of nodes or p nodes in $M_e(v)$ are matches for u' when v is mapped to u. A match in $Q(G)$ must satisfy the topological constraints of Q_π and moreover, the counting quantifiers of Q. Note that the counting quantifier on edge $e = (u, u')$ is applied at *each match* $h_0(u)$ of u, to enforce the semantics of counting.

We denote by $Q(u, G)$ the set of matches of a pattern node u, *i.e.,* nodes $v = h(u)$ induced by all matches h of Q in G. The *query answer* of $Q(x_o)$ in G is defined as $Q(x_o, G)$.

Example 3: For graph G_1 in Fig. 2 and QGP Q_2 of Example 1, $Q_2(x_o, G_1) = \{x_1, x_2\}$. Indeed, 100% of the friends of x_1 and x_2 recommend Redmi 2A. More specifically, for pattern edge $e = \mathsf{follow}(x_o, z)$, when x_o is mapped to x_1 via h_0, $M_e(h_0(x_o), x_1, Q) = \{v_0\}$, which is the set $M_e(x_1)$ of all people whom x_1 follows; similarly when x_o is mapped to x_2. In contrast, while x_3 matches x_o via the stratified pattern of Q_2, $x_3 \notin Q_2(x_o, G_1)$ since at least one user whom x_3 follows (*i.e.,* v_4) has no recom edge to Redmi 2A. □

symbols	notations
$Q(x_o)$	QGP, defined as (V_Q, E_Q, L_Q, f)
$Q(x_o, G)$	query answer, the set of matches of x_o
$Q_\pi(x_o)$	stratified pattern of Q by removing quantifiers
$G' \subseteq G$	G' is a subgraph of G
$M_e(v_x, v, Q)$	$\{v' \mid h \in Q_\pi(G), h(x_o) = v_x, h(e) = (v, v')\}$, $e = (u, u')$
$M_e(v)$	$\{v' \mid (v, v') \in G, L(v, v') = L_Q(e)\}$
$\sigma(e) \odot p\%$	$\frac{\lvert M_e(h_0(x_o), h_0(u), Q)\rvert}{\lvert M_e(h_0(u))\rvert} \odot p\%$, $e = (u, u')$, $h_0 \in Q(G)$
$\sigma(e) \odot p$	$\lvert M_e(h_0(x_o), h_0(u), Q)\rvert \odot p$, $e = (u, u')$, $h_0 \in Q(G)$
$\Pi(Q)$	$Q(x_o)$ excluding nodes with negated edges
Q^{+e}	by positifying a negated edge e in Q
$R(x_o)$	QGAR $Q_1(x_o) \Rightarrow Q_2(x_o)$

Table 1: Notations used in the paper

Negative QGPs. To cope with QGP $Q(x_o)$ with negated edges, we define the following: (1) $\Pi(Q)$: the QGP induced by those nodes in $Q(x_o)$ that are connected to x_o (via a path from or to x_o) with non-negated edges in $Q(x_o)$, *i.e.,* $\Pi(Q)$ excludes all those nodes connected via at least one negated edge; (2) Q^{+e}, obtained by "positifying" a negated edge e in Q, *i.e.,* by changing $f(e)$ from $\sigma(e) = 0$ to $\sigma(e) \geq 1$; and (3) E_Q^-, the set of all negated edges in Q.

Then in a graph G, *query answer* to $Q(x_o)$ is defined as

$$Q(x_o, G) = \Pi(Q)(x_o, G) \setminus \big(\bigcup_{e \in E_Q^-} \Pi(Q^{+e})(x_o, G)\big).$$

That is, we enforce negation via set difference. One can verify that for *each* node u in Q and *each* negated edge $e = (u, u')$ in Q, $|M_e(h_0(x_o), h_0(u), Q)| = 0$.

Example 4: Consider G_1 and Q_3 of Example 1 with p=2. Pattern $\Pi(Q_3)$, which excludes the negated edge $e = (x_o, z_2)$ in Q_3, and $\Pi(Q_3^{+e})$, which"positifies" e in Q_3, are shown in Fig. 3. One can verify the following: (1) $\Pi(Q_3)(x_o, G_1)$ is $\{x_2, x_3\}$; note that x_1 is not a match since only 1 user whom x_1 follows recommends Redmi 2A, and hence violates the counting quantifier $\geq p$; and (2) $\Pi(Q_2^{+e})$ is $\{x_3\}$, which is a "negative" instance for Q_3. Hence, $Q_3(x_o, G_1)$ is $\{x_2\}$, where x_3 is excluded since he follows v_4 who gave a bad rating on Redmi 2A, *i.e.,* violating the negation $\sigma(e) = 0$.

Similarly, for QGP Q_4 and graph G_2 of Fig. 2, when p=2, $Q_4(x_o, G_2)$ is $\{x_5, x_6\}$. Note that node x_4 matches the stratified pattern of Q_4, but it violates the negation on $(x_o$, PhD), which requires that matches of x_o must not be a PhD.

As another example, consider $Q_5(x_o)$ with two negated edges e_1 = (Prof, UK) and e_2 = (z, PhD). It is to find non-UK professors who supervised students who are professors but have no PhD degree. As shown in Fig. 3, $\Pi(Q_5)$ finds professors who supervised students who are professors. In contrast, $\Pi(Q_5^{+e_1})$ finds such professors in the UK, and $\Pi(Q_5^{+e_2})$ (not shown) retrieves professors with students who are professors and have a PhD. In a graph G, $Q(x_o, G) = \Pi(Q_5)(x_o, G) \setminus (\Pi(Q_5^{+e_1})(x_o, G) \cup \Pi(Q_5^{+e_2})(x_o, G))$. □

The notations of this paper are summarized in Table 1.

3. THE COMPLEXITY OF QUANTIFIED MATCHING

In the next three sections, we study *quantified matching*:

- *Input*: A QGP $Q(x_o)$ and a graph G.
- *Output*: $Q(x_o, G)$,

to compute the set of all matches of query focus x_o of Q in G. We start with its complexity in this section.

Decision problem. Its decision problem, referred to as *the quantified matching problem*, is stated as follows.

- *Input*: A QGP $Q(x_o)$, a graph G and a node v in G.

◦ *Question*: Is $v \in Q(x_o, G)$?

When $Q(x_o)$ is a conventional pattern, the problem is NP-complete. When it comes to QGPs, however, ratio aggregates $\sigma \odot p\%$ and negation $\sigma = 0$ increase the expressive power, and make the analysis more intriguing. To handle $\sigma \odot p\%$, for instance, a brute-force approach invokes an NP algorithm that calls a #P oracle to check the ratio aggregate.

We show that while the increased expressive power of QGPs comes with a price, their complexity bound does not get much higher. In particular, #P is not necessary.

Theorem 1: *The quantified matching problem remains* NP*-complete for positive* QGPs*, and it becomes* DP*-complete for (possibly negative)* QGPs. □

Here DP is the class of languages recognized by oracle machines that make a call to an NP oracle and a call to a coNP oracle. That is, L is in DP if there exist languages $L_1 \in$ NP and $L_2 \in$ coNP such that $L = L_1 \cap L_2$ [33].

That is, adding positive quantifiers to conventional graph patterns does not increase the complexity, although ratio aggregates add extra expressive power. Note that such positive patterns alone are already useful in practice. In contrast, the presence of negation makes quantified matching harder, but it remains low in the polynomial hierarchy [33].

The proof is nontrivial. Below we present lemmas needed.

The lower bounds follow from the stronger results below, which are in turn verified by reductions from Subgraph Isomorphism and Exact-Clique, which are NP-complete and DP-complete, respectively (cf. [33]).

Lemma 2: *For* QGPs *with numeric aggregates only, the quantified matching problem is* NP*-hard for positive* QGPs*, and* DP*-hard for (possibly negative)* QGPs. □

The upper bounds are verified by the next two lemmas. In particular, Lemma 4 shows that ratio aggregates can be encoded as numeric aggregates by transforming *both* query Q and graph G, in PTIME. This explains why positive QGPs with ratio aggregates retain the same complexity as conventional patterns, despite their increased expressivity.

Lemma 3: *For* QGPs *with numeric aggregates only, the quantified matching problem is in* NP *for positive* QGPs*, and is in* DP *for (possibly negative)* QGPs. □

Lemma 4: *Any* QGP $Q(x_o)$ *and graph* G *can be transformed in* PTIME *to* QGP $Q_d(x_o)$ *with numeric aggregates only and graph* G_d*, respectively, such that* $Q(x_o, G) = Q_d(x_o, G_d)$. □

Remark. As QGPs with quantifiers bearing $\leq, \neq$ and $<$ subsume the case when $\sigma(e) = 0$, quantified matching is DP-hard for such QGPs. Due to the space limit, we focus on $\geq, =, >$ and leave a full treatment to future work.

4. ALGORITHMS FOR QUANTIFIED MATCHING

We next provide an algorithm, denoted by QMatch, for quantified matching. It takes a QGP $Q(x_o)$ and a graph G as input, and computes $Q(x_o, G)$ as output. It extends existing algorithms $\mathcal{T}$ for conventional subgraph isomorphism, to incorporate quantifier checking and process negated edges.

Generic graph pattern matching. We start by reviewing a generic procedure for subgraph isomorphism, denoted by Match and shown in Fig. 4, slightly adapted from [27] to output $Q(x_o, G)$ for query focus x_o. As observed in [27],

Algorithm Match

Input: pattern $Q(x_o)$, graph G
Output: the answer set $Q(x_o, G)$

1. $Q(x_o, G) := \emptyset$; $Q(G) := \emptyset$; $M := \emptyset$;
2. **for each** u of Q **do**
3. $C(u) :=$ Filtercandidate(Q, G, u);
4. **if** $C(u) = \emptyset$ **then return** $\emptyset$;
5. SubMatch$(Q, G, M, Q(G))$;
6. **for each** isomorphic mapping $h \in Q(G)$ **do**
7. $Q(x_o, G) := Q(x_o, G) \cup \{h(x_o)\}$;
8. **return** $Q(x_o, G)$;

Procedure SubMatch$(Q, G, M, Q(G))$

1. **if** Verify(M) **then**
2. $Q(G) := Q(G) \cup \{h\}$; /*h: the isomorphism defined by M*/
3. **else** $u :=$ SelectNext(Q);
4. **for each** $v \in C(u)$ not matched in M **do**
5. **if** IsExtend(Q, G, M, u, v) **then**
6. $M := M \cup \{(u, v)\}$;
7. SubMatch$(Q, G, M, Q(G))$;
8. Restore(M, u, v);
9. **return**;

Figure 4: Generic search procedure Match

state-of-the-art graph pattern matching algorithms $\mathcal{T}$ typically adopt Match, and differ only in how to optimize key functions (*e.g.,* SelectNext, IsExtend; see below). Given a traditional pattern $Q(x_o)$ and a graph G, Match initializes $Q(x_o, G)$, as well as a partial match M as a set of node pairs (line 1). Each pair (u, v) in M denotes that a node from G matches a pattern node u in Q. It identifies a candidate match set $C(u)$ for each pattern node u in Q (lines 3-4) (FilterCandidate). If there exists a pattern node u with no candidate, it returns $\emptyset$ (line 4). Otherwise, it invokes SubMatch to compute *all matches* (isomorphic mappings) $Q(G)$ (lines 5). It then computes and returns query answer $Q(x_o, G)$ from mappings $h \in Q(G)$ (lines 6-8).

Procedure SubMatch recursively extends partial match M by using three key functions. (1) It picks a pattern node u from Q that has no match yet (SelectNext, line 3). (2) It then checks whether a candidate v of u not yet in M matches u (IsExtend), and if so, it adds (u, v) to M (lines 4-6). (3) It recursively calls SubMatch to extend M with steps (1) and (2) (line 7), and restores it when SubMatch backtracks (line 8). If M is a valid isomorphism (Verify, line 1), it adds M to $Q(G)$ (line 2). This continues until $Q(G)$ is completed.

4.1 Quantified Graph Pattern Matching

Algorithm QMatch revises the generic Match to process quantifiers. (1) It first adopts a *dynamic selection and pruning strategy* to compute $\Pi(Q)(x_o, G)$. The dynamic search picks top p promising neighbors based on a potential score, with p adapted to the corresponding quantifiers. (2) It then employs *optimal incremental evaluation* to process negated edges, which maximally reuses cached matches for $\Pi(Q)$ when processing Q^{+e} for positified e, instead of recomputing $Q^{+e}(G)$ starting from scratch. The strategies are supported by optimized data structures and key functions from Match.

Auxiliary structures. QMatch maintains auxiliary structures for each node v in $C(u)$ as follows: (1) a Boolean variable $X(u, v)$ indicating whether v is a match of u via isomorphism from $\Pi(Q)$ to G, and (2) a vector T, where entry $T(v, e)$ for an edge $e=(u, u')$ in Q is a pair $\langle c(v, e), U(v, e)\rangle$, in which c (resp. U, initialized as $M_e(v)$) records the current size (resp. an estimate upper bound for) $|M_e(v_x, v, Q)|$.

Algorithm QMatch

Input: a QGP $Q(x_o)$, graph G
Output: the answer set $Q(x_o, G)$

1. $Q(x_o, G)$:= $\emptyset$; M:=$\emptyset$; $\Pi(Q)(x_o, G)$:=$\emptyset$; $\Pi(Q)(G)$:=$\emptyset$;
2. **for each** u of Q **do**
3. initializes $C(u)$ and auxiliary structures;
4. $\Pi(Q)(x_o, G)$:=DMatch$(\Pi(Q), G, M, \Pi(Q)(G))$;
5. **for each** negative edge e in E_Q^- **do**
6. $Q^{+e}(x_o, G)$:=IncQMatch$(\Pi(Q)(x_o, G), Q^{+e})$;
7. $Q(x_o, G)$:=$\Pi(Q)(x_o, G) \setminus \bigcup_{e \in E_Q^-} Q^{+e}(x_o, G)$;
8. **return** $Q(x_o, G)$;

Figure 5: Algorithm QMatch

Algorithm. Algorithm QMatch (outlined in Fig. 5) revises Match to process QGP $Q(x_o)$ in three steps. (1) It first initializes the candidate set and auxiliary structures with a revised Filtercandidate (lines 1-3). For each pattern node u in $Q(x_o)$, it initializes (a) $C(u)$ with nodes v of the same label, and (b) $X(u, v) = \perp$, $c(v, e)$=0 and $U(v, e) = |M_e(v)|$ for each e=(u, u') in Q. It removes v from $C(u)$ if $U(v, e)$ does not satisfy the quantifier of e. (2) It next invokes a procedure DMatch revised from SubMatch in Fig. 4 to compute $\Pi(Q)(x_o, G)$ (line 4). (3) It then processes each negated edge e by constructing its positified pattern Q^{+e}, and computes $Q^{+e}(x_o, G)$ with an incremental procedure IncQMatch (lines 5-6). (4) It computes $Q(x_o, G)$ by definition (line 7). We next present DMatch, and defer IncQMatch to Section 4.2.

Example 5: Given Q_3 with p=2 (Fig. 1) and G_1 (Fig. 2), QMatch first computes $\Pi(Q_3)(x_o, G_1)$ (Fig. 3). It initializes variables for nodes in G_1, partially shown below ($i \in [0, 4]$).

	X	c	U
x_1	$X(x_o, x_1)$=$\perp$	$c(x_1, (x_o, z_1))$=0	$U(x_1, (x_o, z_1))$=1
x_2	$X(x_o, x_2)$=$\perp$	$c(x_2, (x_o, z_1))$=0	$U(x_2, (x_o, z_1))$=2
x_3	$X(x_o, x_3)$=$\perp$	$c(x_3, (x_o, z_1))$=0	$U(x_3, (x_o, z_1))$=3
v_i	$X(x_o, v_i)$=$\perp$	$c(v_i, (z_1$,Redmi$))$=0	$U(v_i, (z_1$, Redmi$))$=1

At this stage, since $U(x_1, (x_o, z_1))$=1 $\leq$ 2, x_1 fails the quantifier of (x_o, z_1), and is removed from $C(x_o)$. □

Procedure DMatch. Given positive QGP $\Pi(Q)$, DMatch revises SubMatch (Fig 4) by adopting dynamic search. To simplify the discussion, we consider numeric $\sigma(e) \odot p$ first.

(1) Given a selected pattern node u' (line 3 of SubMatch), a candidate $v \in C(u)$, and an edge e=(u, u') with quantifier $\sigma(e) \odot p$, DMatch *dynamically* finds p best nodes (recorded in a heap $S_P(u')$) from $C(u')$ that are children of v (lines 4-5 of SubMatch, IsExtend), using selection and pruning rules (see Appendix B). Denote as $P(v')$ the parent set of v' in G, the *potential* of a match $v' \in C(u')$ is defined as:

$$(1 + \frac{|P(v') \cap C(u)|}{|C(u)|}) * \Sigma_{\forall e=(u', u'')} \frac{U(v', e)}{p_e},$$

where p_e is the number in $\sigma(e) \odot p_e$ for edge e=(u', u''). It favors those candidates that (a) benefit the verification of more candidates during future backtracking, and (b) have high upper bounds *w.r.t.* p (hence more likely to be a match itself). We select candidates with the highest scores.

DMatch then updates M by including (u, v), and recursively conducts the next level of search by forking p verifications in the order of the selected p candidates (line 7, SubMatch). It keeps a record of M and a cursor to memorize the candidates in S_P for backtracking, using a stack.

(2) When backtracking to a candidate $v \in S_P(u)$ from a child v' of v, DMatch restores M and the cursor (Restore, line 8 of SubMatch). It next dynamically updates $S_P(u)$. (a) If $X(u', v')$=false, it reduces $U(v, e)$ by 1. (b) It applies the selection and pruning rules to $C(u)$ using the *updated* potentials *w.r.t.* the changes in (a). If the upper bound $U(v, e)$ fails the quantifier of e, v is removed from $C(u)$ and $S_P(u)$ *without* further verifying its other children. Otherwise, it picks a new set $S_P(u)$ of candidates with top potentials.

(3) When M is complete, *i.e.,* each node u in $\Pi(Q)$ has a match in M, DMatch checks whether M is an isomorphic mapping. If so, it updates $X(u, v)$=true for each pair $(u, v) \in M$, and increases the counter $c(v, e)$. It then checks whether the counters *satisfy the quantifiers* of $\Pi(Q)$. If so, it adds v_x to $Q(x_o, G)$. Otherwise, it proceeds. DMatch terminates when all the candidates of x_o are checked.

Example 6: Continuing with Example 5, given $C(x_o)$ = $\{x_2, x_3\}$, DMatch selects x_2, and extends M with (x_o, x_2). In contrast to Fig 4, DMatch picks top 2 best candidates $S_P(z_1)$ = $\{v_2, v_1\}$ from $C(z_1)$ following edge e=(x_o, z_1). This adds (z_1, v_2) to M, and (Redmi 2A, Redmi 2A) for the next round. At verification, it finds M a complete isomorphism, and updates $X(v_o, x_2)$=true and $c(x_2, e)$=1. As x_2 cannot be verified as a match via $\Pi(Q_3)$ yet, DMatch next verifies v_1, and sets $c(x_2, e)$=2. As x_2 is a match and has a counter satisfying the quantifier, it is added to $\Pi(Q_3)(x_o, G_1)$. The updated variables for candidates of $C(x_o)$ are as follows.

	X	c	U
x_2	$X(x_o, x_2)$=True	$c(x_2, (x_o, z_1))$=2	$U(x_2, (x_o, z_1))$=2
x_3	$X(x_o, x_3)$=$\perp$	$c(x_3, (x_o, z_1))$=2	$U(x_3, (x_o, z_1))$=3

DMatch next verifies x_3. It starts by selecting top 2 candidates $S_P(x_3)$=$\{v_2, v_3\}$. Once v_3 is processed, it finds that x_3 is in an isomorphism with $c(x_3, e)$=2, and hence is a match. It returns $\{x_2, x_3\}$ as $\Pi(Q_3)(x_o, G_1)$. □

One can readily verify the following (see Appendix A).

Lemma 5: DMatch *computes* $\Pi(Q)(x_o, G)$ *by (a) verifying no more candidates than any* Match*-based subgraph isomorphism algorithm* $\mathcal{T}$*, and (b) with space cost* $O(p_m|Q| + |V|)$*, where* p_m *is the largest constant in all quantifiers of* Q. □

That is, quantified matching can be evaluated following conventional $\mathcal{T}$ without incurring significant extra time and space cost. The performance of DMatch is further improved by selection and pruning rules presented in Appendix B.

Ratio aggregates. DMatch can be readily extended to process ratio aggregates. Indeed, for each pattern e=(u, u') with $\sigma(e) \odot p\%$ and at a candidate v of u, DMatch "transforms" the quantifier to its equivalent numeric counterpart $\sigma(e) \odot p'$ as follows. (a) DMatch computes $|M_e(v)|$ by definition. (2) It sets $p' = \lfloor |M_e(v)| * p\% \rfloor$. The transformation for e preserves all the exact matches for ratio quantifiers by definition, and takes a linear scan of G (in $O(|G|)$ time). In addition, QMatch easily extends to QGPs with quantifiers $\sigma(e) > p$, by replacing it with $\sigma(e) \geq p + 1$.

4.2 Incremental Quantified Matching

If $\Pi(Q)(x_o, G)$ is nonempty, QMatch proceeds to compute $\Pi(Q^{+e})(x_o, G)$ for each negated edge $e \in E_Q^-$ (lines 5-6, Fig. 5). Observe the following: (1) $\Pi(Q^{+e})$=$\Pi(Q) \oplus \Delta E$, *i.e.,* $\Pi(Q^{+e})$ "expands" $\Pi(Q)$ with a set ΔE of positive edges; and (2) for any node u in $\Pi(Q)$, $\Pi(Q^{+e})(u, G) \subseteq \Pi(Q)(u, G)$, since $\Pi(Q^{+e})$ adds more constraints to $\Pi(Q)$.

This observation motivates us to study a novel *incremental quantified matching* problem. Given a graph G, a QGP Q,

computed matches $Q(u, G)$ for each u in Q, and a new QGP Q'=$Q \oplus \Delta E$, it is to compute $Q'(x_o, G) = Q(x_o, G) \oplus \Delta O$, *i.e.,* to find changes ΔO in the output. It aims to make maximum use of cached results $Q(u, G)$, instead of computing $Q'(x_o, G)$ from scratch. As opposed to conventional incremental problems [15, 34], we compute ΔO in response to changes in *query* Q, rather than to changes in graph G.

As observed in [34], the complexity of incremental graph problems should be measured in the size of *affected area*, which indicates the amount of work that is necessarily performed by any algorithm for the incremental problem. For pattern matching via subgraph isomorphism, the number of verifications is typically the major bottleneck. Below we identify affected area for quantified matching, to characterize the optimality of incremental quantified matching.

Optimal incremental quantified matching. Given Q and $\Pi(Q^{+e})$, the *affected area* is defined as

$$\mathsf{AFF} = \bigcup C(u_i) \cup \{N(v) \mid v \in C(u_i)\},$$

where (1) u_i is in edge e_i=(u_i, u_i') or (u_i', u_i) for each $e_i \in \Delta E$; (2) $C(u_i)$ includes (a) the match sets cached after DMatch processed $\Pi(Q)$; and (b) the candidate sets initialized by QMatch (line 3 of Fig 5) for new nodes u_i introduced by $\Pi(Q^{+e})$, which have to be checked; and (c) $N(v)$ is the set of nodes in cached $C(\cdot)$ that are reachable from (or reached by) v, via paths that contains only the nodes in $C(\cdot)$.

An incremental quantified matching algorithm is *optimal* if it incurs $O(|\mathsf{AFF}|)$ number of verifications. Intuitively, AFF is the set of nodes that are necessarily verified in response to ΔE, for any such algorithms to find exact matches.

Proposition 6: *There exists an incremental algorithm that computes each $\Pi(Q^{+e})(x_o, G)$ by conducting at most $|\mathsf{AFF}|$ rounds of verification.* □

As a proof, we present an optimal algorithm IncQMatch.

Procedure IncQMatch. Algorithm IncQMatch (used in line 6, Fig. 5) incrementally computes $\Pi(Q^{+e})(x_o, G)$ by reusing the cached match sets and the counters computed in the process of DMatch for $\Pi(Q)$. It works as follows.

(1) IncQMatch initializes $\Pi(Q^{+e})(u, G)$ for each u with the cached matches $\Pi(Q)(u, G)$. It then computes the edge set ΔE in $\Pi(Q^{+e})(x_o, G)$ to be "inserted" into $\Pi(Q)$.

(2) IncQMatch then iteratively processes the edges e=(u, u') in ΔE. It first identifies those cached matches that are *affected* by the insertion. It considers two possible cases below.

- Both u and u' are in $\Pi(Q)$. For each match $v \in \Pi(Q)(u, G)$, IncQMatch adds v to AFF and verifies whether v matches u via isomorphism following DMatch. If $X(u, v)$=true, it counts $c(v, e)$ as the number of v's children v' that are matches of u' and checks the quantifier of e. If $c(v, e)$ satisfies the quantifier, no change happens. Otherwise ($c(v, e)$ fails the quantifier) IncQMatch removes v from $\Pi(Q^{+e})(u, G)$.
- One or both of nodes u and u' are not in $\Pi(Q)$. For the new node u (or u'), IncQMatch treats e as a single edge pattern and verifies each candidate v_1 in $C(u)$. Since $\Pi(Q)$ is a "sub-pattern" of $\Pi(Q^{+e})$ and $\Pi(Q^{+e})$ is connected, we need only to inspect those matches v_1 reachable from some nodes in cached $C(.)$, *i.e.,* $v_1 \in N(v_2)$ for some cached v_2; hence $v_1 \in$ AFF.

For each v removed in the steps above, QMatch then propagates the impact recursively by (a) reducing all the counters of v's parents by 1, and (b) removing invalid matches due to the updated counters and adds them to AFF, following the same backtracking verification as in DMatch, until a fixpoint is reached, *i.e.,* no more matches can be removed.

Example 7: Continuing with Example 6, QMatch invokes IncQMatch to process $\Pi(Q_3^{+(x_o, z_2)})$, with $\Delta E = \{(x_o, z_2), (z_2, \text{Redmi 2A})\}$ (see Fig. 3) as follows.

(1) IncQMatch first initializes the candidate sets as the cached matches in DMatch (shown below). For node z_2 not in $\Pi(Q)$, IncQMatch finds $C(z_2)$ as initialized in QMatch.

pattern node	$C(\cdot)$
x_o	$C(x_o)$=$\{x_2, x_3\}$
z_1	$C(z_1)$=$\{v_1, v_2, v_3\}$
z_2	$C(z_2)$=$\{v_4\}$
Redmi	C(Redmi 2A)=$\{$Redmi 2A$\}$

(2) It starts with edge (z_2, Redmi 2A), and initializes AFF as $C(z_2) \cup C(\text{Redmi 2A})$=$\{v_4$, Redmi 2A$\}$. It next checks whether v_4 and Redmi 2A remain matches with counter satisfying the quantifiers. In this process, it only visits the two cached matches x_3 and v_3 following the pattern edges. As both nodes are matches, no change needs to be made.

(3) IncQMatch next processes edge (x_o, z_2). It adds the set $C(x_o)$=$\{x_2, x_3\}$ to AFF, and checks whether x_2 and x_3 remain matches. As x_2 has no edge to v_4, $X(x_o, x_2)$ is updated to false, and x_2 is removed from $C(x_o)$. It next finds that x_3 is a valid match, by visiting v_2, v_3, Redmi 2A, and v_4.

As no more matches can be removed, IncQMatch stops the verification. It returns $\Pi(Q_3^{+(x_o, z_2)})(x_o, G_1)$ as $\{x_3\}$. After the process, AFF contains $\{v_4, x_2, x_3, v_2, v_3$, Redmi 2A$\}$. It incurs in total 3 rounds of verification for v_4, x_2 and x_3. □

Contrast IncQMatch with DMatch. (1) IncQMatch only visits the cached matches and their edges, rather than the entire G. (2) IncQMatch incurs at most $|\mathsf{AFF}|$ rounds of verifications; hence it is *optimal w.r.t.* incremental complexity.

Analysis of QMatch. Algorithm QMatch correctly computes $Q(x_o, G)$ following the definition of quantified matching (Section 2.2). For its complexity, observe the following.

(1) If Q is positive, *i.e.,* $E_Q^- = \emptyset$, IncQMatch is not needed. Then QMatch and a conventional Match-based algorithm $\mathcal{T}$ for subgraph isomorphism have the same complexity. Quantifier checking is incorporated into the search process.

(2) If E_Q^- is nonempty, IncQMatch invokes at most $|E_Q^-|$ rounds of incremental computation by *optimal* IncQMatch, while $|E_Q^-| \leq |Q|$ and Q is typically small in practice. For each round, the overall time taken is bounded by $|\mathsf{AFF}| * K$, where K is the cost of a single verification.

Put together, QMatch takes $O(t(\mathcal{T}) + |E_Q^-||\mathsf{AFF}| * K)$ time in total, where $t(\mathcal{T})$ is the time complexity of a Match-based algorithm $\mathcal{T}$ for conventional subgraph isomorphism. We find in our experiments that QMatch and $\mathcal{T}$ have comparable performance, due to small $|E_Q^-|$ and $|\mathsf{AFF}|$. Moreover, existing optimization for $\mathcal{T}$ can be readily applied to QMatch.

Algorithm QMatch also makes use of graph simulation [21] to filter candidates and reduce verification cost. We defer this optimization strategy to Appendix-B.

5. PARALLEL QUANTIFIED MATCHING

Quantified matching – in fact even conventional subgraph isomorphism – may be cost-prohibitive over big graphs G.

This suggests that we develop a parallel algorithm for quantified matching that guarantees to scale with big G. We develop such an algorithm, which makes quantified matching feasible in real-life graphs, despite its DP complexity.

5.1 Parallel Scalability

To characterize the effectiveness of parallelism, we advocate a notion of *parallel scalability* following [16, 26]. Consider a problem I posed on a graph G. We denote by $t(|I|, |G|)$ the running time of the best *sequential algorithm* for solving I on G, *i.e.,* one with the least worst-case complexity among all algorithms for I. For a parallel algorithm, we denote by $T(|I|, |G|, n)$ the time it takes to solve I on G by using n processors, taking n as a parameter. Here we assume $n \ll |G|$, *i.e.,* the number of processors does not exceed the size of G; this typically holds in practice as G often has trillions of nodes and edges, much larger than n [20].

Parallel scalability. An algorithm is *parallel scalable* if

$$T(|I|, |G|, n) = O(\frac{t(|I|, |G|)}{n}) + (n|I|)^{O(1)}.$$

That is, the parallel algorithm achieves a linear reduction in sequential running time, plus a "bookkeeping" cost $O((n|I|)^l)$ that is *independent of* $|G|$, for a constant l.

A parallel scalable algorithm *guarantees* that the more processors are used, the less time it takes to solve I on G. Hence given a big graph G, it is feasible to efficiently process I over G by adding processors when needed.

5.2 Parallel Scalable Algorithm

Parallel scalability is within reach for quantified matching under certain condition. We first present some notations. For a node v in graph G and an integer d, the *d-hop neighbor* $N_d(v)$ of v is defined as the subgraph of G induced by the nodes within d hops of v. The *radius* of a QGP $Q(x_o)$ is the longest shortest distance between x_o and any node in Q.

The main result of the section is as follows.

Theorem 7: *There exists an algorithm* PQMatch *that given* QGP $Q(x_o)$ *and graph* G*, computes* $Q(x_o, G)$*. It is parallel scalable for graphs* G *with* $\sum_{v \in G} |N_d(v)| \leq C_d * \frac{|G|}{n}$*, taking* $O(\frac{t(Q,G)}{n} + n)$ *time, where* d *is the radius of* $Q(x_o)$*,* C_d *is a predefined constant, and* $t(Q, G)$ *is the worst-case running time of sequential quantified matching algorithms.* □

The condition is practical: 99% of real-life patterns have radius at most 2 [18], and the average node degree is 14.3 in social graphs [12]; thus $|N_d(v)|$ is often a small constant. In addition, we will show that PQMatch can be adapted to evaluate QGPs Q with radius larger than d.

As a proof, below we present PQMatch. The algorithm works with a coordinator S_c and n workers (processors) S_i. It utilizes two levels of parallelism. (a) At the *inter-fragment parallelism* level, it creates a partition scheme of G over multiple processors *once for all*, so that quantified matching is performed on all these fragments in parallel. The same partition is used for all QGPs $Q(x_0)$ within radius d. (b) At the *intra-fragment* level, local matching within each fragment is further conducted by multiple threads in parallel.

Hop preserving partition. We start with graph partition. To maximize parallelism, a partition scheme should guarantee that for any graph G, (1) each of n processors manages a small fragment of approximately equal size, and (2) a query can be evaluated locally at each fragment without incurring inter-fragment communication. We propose such a scheme.

Given a graph $G = (V, E, L)$, an integer d and a node set $V' \subseteq V$, a *d-hop preserving partition* $\mathcal{P}_d(V')$ of V' distributes G to a set of n processors such that it is

(1) *balanced*: each processor S_i manages a fragment F_i, which contains the subgraph G_i of G induced by a set V_i of nodes, such that $\bigcup V_i = V'$ ($i \in [1, n]$) and the size of F_i is bounded by $c * \frac{|G|}{n}$, for a small constant $c < C_d$; and

(2) *covering*: each node $v \in V'$ is *covered* by $\mathcal{P}_d(V')$, *i.e.,* there exists a fragment F_i such that $N_d(v)$ is in F_i.

We say that $\mathcal{P}_d(V')$ is *complete* if $|V'| = |V|$.

One naturally wants to find an optimal partition such that the number $|V'|$ of covered nodes is maximized. Although desirable, creating a balanced d-hop preserving partition is NP-hard. Indeed, conventional balanced graph partition is a special case when d=1, which is already NP-hard [6].

Parallel d-hop preserving partition. We provide an approximation algorithm for d-hop preserving partition with an approximation ratio. Better still, it is parallel scalable.

Lemma 8: *If* $\sum_{v \in G} |N_d(v)| \leq C_d * \frac{|G|}{n}$*, for any constant* $\epsilon > 0$*, there is a parallel scalable algorithm with approximation ratio* $1 + \epsilon$ *to compute a d-hop preserving partition.* □

Below we present such an algorithm, denoted by DPar. Given a graph G stored at the coordinator S_c, it starts with a base partition of G, where each fragment F_i has a balanced size bounded by $c * \frac{|G|}{n}$. This can be done by using an existing balanced graph partition strategy (*e.g.,* [23]). DPar then extends each fragment F_i to a d-hop preserving counterpart.

(1) It first finds the "border nodes" $F_i.O$ of F_i that have d-hop neighbors not residing in F_i, by traversing F_i in parallel.

(2) Each worker S_i then computes and loads $N_d(v)$ for each $v \in F_i.O$, by "traversing" G via disk-based parallel breadth-first search (BFS) search [24]. Moreover, DPar uses a balanced loading strategy (see below) to load approximately equal amount of data to each worker in the search. The process repeats until no fragments can be expanded.

Balancing strategy. DPar enforces a balanced fragment size $c * \frac{|G|}{n}$. It conducts a d-hop preserving partition $\mathcal{P}_d(V')$ with approximation ratio $1 - \epsilon$ subject to the bound, for any given ϵ. That is, if the size of nodes covered by the optimal d-hop partition in G is $|V^*|$, then $\mathcal{P}_d(V')$ has $|V'| \geq (1 - \epsilon)|V^*|$.

More specifically, at the BFS phase, for each $v \in \bigcup F_i.O$, DPar assigns $N_d(v)$'s to workers by reduction to *Multiple Knapsack problem* (MKP) [13]. Given a set of weighted items (with a value) and a set of knapsack with capacities, MKP is to assign each item to a knapsack subject to its capacity, such that the total value is maximized. DPar treats each $N_d(v)$ as an item with value 1 and weight $|N_d(v)|$, and each fragment as a knapsack with capacity $c * \frac{|G|}{n} - |F_i|$, with the number of covered nodes as the total value. It solves the MKP instance by invoking the algorithm of [13], which computes an assignment with approximation ratio $1 + \epsilon$ for any given ϵ, in $O(|V'|^{\frac{1}{\epsilon}})$ time. Each worker S_i then loads its assigned $N_d(v)$. This gives us a d-hop preserving partition $\mathcal{P}_d$ with ratio $1 + \epsilon$ (see Appendix A for the reduction).

Partition $\mathcal{P}_d$ may not be complete, *i.e.,* not every node in V is covered. To maximize inter-fragment parallelism, DPar "completes" $\mathcal{P}_d$ while preserving the balanced partition size.

Algorithm PQMatch

Input: QGP $Q(x_o)$, graph G, coordinator S_c, n workers $S_1, \ldots, S_n$
Output: the answer set $Q(x_o, G)$.

1. DPar(G); /*Preprocessing*/
/*executed at coordinator S_c*/
2. $Q(x_o, G) := \emptyset$; post Q to each worker;
3. **if** every worker S_i returns answer $Q(x_o, F_i)$ **then**
4. $Q(x_o, G) := \bigcup Q(x_o, F_i)$;
5. **return** $Q(x_o, G)$;
/*executed at each worker in parallel*/
6. $Q(x_o, F_i) :=$ mQMatch(b, Q, F_i); /* b: the # of threads*/
7. **return** $Q(x_o, F_i)$;

Figure 6: Algorithm PQMatch

For each uncovered node v, it assigns $N_d(v)$ to a worker S_i that minimizes estimated size difference $|F_{max}| - |F_{min}|$, where F_{max} (resp. F_{min}) is the largest (resp. smallest) fragment *if* $N_d(v)$ is merged to F_i. Since $\sum |N_d(v)| \leq C_d * \frac{|G|}{n}$, this suffices to make $\mathcal{P}_d$ both complete and d-preserving.

Example 8: Consider graph G_2 of Fig. 2 and a set $V' = \{v_5, \ldots, v_9\}$. Assume a base partition distributes $\{v_5\}$ to worker S_1, $\{v_7, v_9\}$ to S_2, $\{v_6, v_8\}$ to S_3, respectively. DPar creates a 1-hop preserving partition $\mathcal{P}_1$ for V' as follows. (1) Each S_i identifies its border nodes $F_i.O$ by a local traversal, *e.g.,* $v_5 \in F_1.O$. (2) Each S_i traverses G_2 at S_c and finds $N_1(v)$ for $v \in F_i.O$, in parallel. At the end, S_c keeps track of the nodes (edges) "requested" by workers as follows.

node	requested by
x_4	S_1, S_3
x_5, x_6, **prof.**	S_2, S_3
PhD.	S_1, S_2, S_3

DPar next determines which site to send the border nodes by solving an MKP instance, shown as follows.

site	$N_1(\cdot)$	(estimated) $\lvert F_i \rvert$
S_1	$N_1(v_5)$, $N_1(v_9)$	14 ($\lvert N_1(v_5) \rvert = 6$, $\lvert N_1(v_9) \rvert = 8$)
S_2	$N_1(v_7)$	8
S_3	$N_1(v_8)$	15

Here $N_1(v_5)$ includes three nodes x_4, v_5, PhD, and three edges (x_4, v_5), $(v_5$, PhD) and $(x_4$, PhD); similarly for the others. This induces a 1-hop preserving partition $\mathcal{P}_1$ that covers $\{v_5, v_7, v_8, v_9\}$. To complete $\mathcal{P}_1$, DPar selects S_2 to load $N_1(v_6)$, where $|N_1(v_6)| = 15$. This minimizes the estimated size $|F_{max}| - |F_{min}| = 19 - 14 = 5$. Here $|F_{max}|$ is estimated as the sum of $|F_2| = 8$ and 11 additional nodes and edges in $N_1(v_6)$ that are not "requested" by S_2 (*e.g.,* (x_4, v_6), $(v_6$, PhD)). The completed $\mathcal{P}_1$ covers V' with fragment size 14, 19 and 15 for S_1, S_2 and S_3, respectively. □

Parallel algorithm. Using DPar, we next develop algorithm PQMatch. As shown in Fig. 6, PQMatch takes as input a QGP $Q(x_o)$ of radius at most d, and a graph G distributed across n workers by DPar, where fragment F_i of G resides at worker S_i. It works as follows. (1) The coordinator S_c posts $Q(x_o)$ to each worker S_i (line 2). (2) Each worker S_i then invokes a procedure mQMatch to compute local matches $Q(x_o, F_i)$ (line 7), where mQMatch implements QMatch using multi-threading (see below). Once verified, $Q(x_o, F_i)$ is sent to S_c (line 6). (3) Once all the workers have sent their partial matches to S_c, the coordinator computes $Q(x_o, G)$ as the union of all $Q(x_o, F_i)$ (lines 3-4).

Procedure mQMatch. Procedure mQMatch is a multi-threading implementation of PQMatch (Section 4), supporting inter-fragment level parallelism. For pattern edge $e = (u, u')$ with quantifier $\sigma(e) \odot p$ and a candidate v in $C(u)$, it spawns p threads to simultaneously verify the top p selected candidates, one for each. Each thread i maintains local partial matches (in its local memory). When all the p threads backtrack to v, the local partial matches are merged, and the local counter of u is updated by aggregating the local storage of each thread i (see Appendix B for more details).

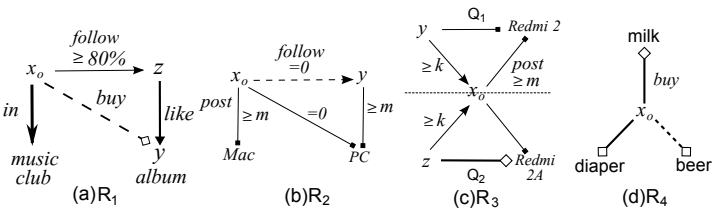


Figure 7: QGARs

From Lemma 8 and Lemma 9 below, Theorem 7 follows (see Appendix A for a proof). We remark that G is partitioned *once* by using a d-hop preserving partition process. Then for all QGPs with radius within d, *no re-partitioning* is needed. That is, condition $\sum |N_d(v)| \leq C_d * \frac{|G|}{n}$ is needed only for d-hop preserving partition to be parallel scalable.

Lemma 9: *Given G distributed over n processors by a d-hop preserving partition $\mathcal{P}_d$, (1) $Q(x_o, G) = \bigcup_{i \in [1,n]} Q(x_o, F_i)$, and (2)* mQMatch *is parallel scalable for all* QGPs *$Q(x_o)$ with radius bounded by d.* □

Remark. Algorithm PQMatch can be easily adapted to dynamic query load and graphs. (1) For a query with radius $d' > d$, each worker S_i incrementally computes $N_{d'-d}(v)$ for each node $v \in F_i.O$, via the balanced parallel BFS traversal. (2) When G is updated, coordinator S_c assigns the changes (*e.g.,* node/edge insertions and deletions) to each fragment. Each worker then applies incremental distance querying [15] to maintain $N_d(v)$ of all affected $v \in F_i.O$ for $i \in [1, n]$.

6. QUANTIFIED ASSOCIATION RULES

As an application of QGPs, we introduce a set of graph association rules (QGARs) with counting quantifiers, to identify regularity between entities in graphs in general, and potential customers in social graphs in particular.

QGARs. A *quantified graph association rule* $R(x_o)$ is defined as $Q_1(x_o) \Rightarrow Q_2(x_o)$, where Q_1 and Q_2 are QGPs, referred to as the *antecedent* and *consequent* of R, respectively.

The rule states that for all nodes v_x in a graph G, if $v_x \in Q_1(x_o, G)$, then the chances are that $v_x \in Q_2(x_o, G)$.

Using QGPs, QGAR R can express *positive and negative* correlations [40] and social influence patterns with statistical significance [19], which are useful in targeted advertising. (1) If Q_2 is a positive QGP, $R(x_o)$ states that if x_o satisfies the conditions in Q_1, then "event" Q_2 is likely to happen to x_o. For instance, $Q_2(x_o)$ may be a single edge buy(x_o, y) indicating that x_o may buy product y. In a social graph G, $R(x_o, G)$ identifies potential customers x_o of y. (2) When Q_2 is, *e.g.,* a single negated edge buy(x_o, y), $R(x_o)$ suggests that no v_x in $Q_1(x_o, G)$ will likely buy product y.

Example 9: A positive QGAR $R_1(x_o)$: $Q_1(x_o) \Rightarrow$ buy(x_o) is shown in Fig. 7, where Q_1 is the QGP given in Example 1, and Q_2 is a single edge buy(x_o) (depicted as a dashed edge). It states that if x_o is in a music club and if 80% of people whom x_o follows like an album y, then x_o will likely buy y.

A negative QGAR R_2 is also shown in Fig. 7, where Q_2 is a single negative edge $\mathsf{follow}(x_o, y)$. The QGAR states that if x_o and y actively ($\geq k$) tweet on competitive products (*e.g.,* "Mac" vs "PC"), then x_o is unlikely to follow y. Intuitively, R_2 demonstrates "negative" social influence [19].

As another example, R_3 of Fig. 7 is a rule in which Q_2 consists of multiple nodes. Here Q_1 in R_3 specifies users x_o who actively promote mobile phone Redmi 2 and influence other users; and Q_2 predicts the impact of x_o on other users for a new release Redmi 2A. Putting these together, R_3 states that if x_o is influential over an earlier version, then x_o is likely to promote the selling of a new release [4]. Intuitively, Q_1 identifies x_o as "leaders" [19], who are often targeted by companies for promotion of a product series [4].

To the best of our knowledge, these QGARs are not expressible as association rules studied so far (*e.g.,* [16, 17]).

QGARs also naturally express conventional association rules defined on itemsets. For instance, $\mathsf{milk}, \mathsf{diaper} \Rightarrow \mathsf{beer}$ is depicted as QGAR $R_4(x_o)$ in Fig. 7. It finds customers x_o who, if buy milk and diaper, are likely to purchase beer. □

For real-world applications (*e.g.,* social recommendation), we consider practical and nontrivial QGARs by requiring: (a) Q_1 and Q_2 are connected and *nonempty* (*i.e.,* each of them has at least one edge); and (b) Q_1 and Q_2 do not overlap, *i.e.,* they do not share a common edge. We treat R as a QGP composed of both Q_1 and Q_2 such that in a graph G,

$$R(x_o, G) = Q_1(x_o, G) \cap Q_2(x_o, G).$$

Interestingness measure. To identify interesting QGARs, we define the support and confidence of QGARs.

Support. Given a QGAR $R(x_o)$ and a graph G, the *support* of R in G, denoted as $\mathsf{supp}(R, G)$, is the size $|R(x_o, G)|$, *i.e.,* the number of matches in $Q_1(x_o, G) \cap Q_2(x_o, G)$. We justify the support with the result below, which shows its *anti-monotonicity* for both pattern topology *and* quantifiers.

Lemma 10: *For any extension R' of R by (1) adding new edges (positive or negative) to Q_1 or Q_2, or (2) increasing p in positive quantifiers, $|\mathsf{supp}(R', G)| \leq |\mathsf{supp}(R, G)|$.* □

Confidence. We follow the local close world assumption (LCWA) [14], assuming that graph G is locally complete, *i.e.,* either G includes the complete neighbors of a node for any known edge type, or it has no information about these neighbors. We define the confidence of $R(x_o)$ in G as

$$\mathsf{conf}(R, G) = \frac{|R(x_o, G)|}{|Q_1(x_o, G) \cap X_o|},$$

where X_o is the set of candidates of x_o that are associated with an edge of the same type for every edge $e{=}(x_o, u)$ in Q_2. Intuitively, X_o retains those "true" negative examples under LCWA, *i.e.,* those that have every required relationship of x_o in Q_2 but are not a match (see Appendix C for justification).

Quantified entity identification. We want to use QGARs to identify entities of interests that match certain behavior patterns specified by QGPs. To this end, we define the set of entities identified by a QGAR $R(x_o)$ in a (social or knowledge) graph G with confidence η as follows:

$$R(x_o, \eta, G) = \{v_x \mid v_x \in R(x_o, G), \mathsf{conf}(R, G) \geq \eta\},$$

i.e., entities identified by R if its confidence is above η.

We study the *quantified entity identification* (QEI) problem: Given a QGAR $R(x_o)$, graph G, and a confidence threshold $\eta > 0$, it is to find all the entities in $R(x_o, \eta, G)$.

The QEI problem is DP-hard, as it embeds the quantified matching problem, which is DP-hard (Theorem 1). However, the (parallel) quantified matching algorithms for QGPs can be extended to QEI, *without* incurring substantial extra cost. Denote as $t(|Q|, |G|)$ the cost for quantified matching of QGP Q in G. Then we have the following (Appendix A).

Corollary 11: *There exist (1) an algorithm to compute $R(x_o, \eta, G)$ in $O(t(|R|, |G|))$ time; and (2) a parallel scalable algorithm to compute $R(x_o, \eta, G)$ in $O(\frac{t(|R|,|G|)}{n} + n)$ time with n processors, under the condition of Theorem 7.* □

7. EXPERIMENTAL STUDY

We conducted three sets of experiments to evaluate (1) the scalability and (2) parallel scalability of our quantified matching algorithms, and (3) the effectiveness of QGAR for identifying correlated entities in large real-world graphs.

Experimental setting. We used two real-life graphs: (a) *Pokec* [2], a social network with 1.63 million nodes of 269 different types, and 30.6 million edges of 11 types, such as *follow*, *like*; and (b) *YAGO2*, an extended knowledge base of YAGO [39] that consists of 1.99 million nodes of 13 different types, and 5.65 million links of 36 types.

We also developed a generator to produce synthetic social graphs $G = (V, E, L)$, controlled by the numbers of nodes $|V|$ (up to 50 million) and edges $|E|$ (up to 100 million), with L drawn from an alphabet $\mathcal{L}$ of 30 labels. The generator is based on GTgraph [7] following the small-world model.

Pattern generator. For real-life graphs we generated QGPs Q controlled by $|V_Q|$ (size of pattern nodes), $|E_Q|$ (pattern edges), p% (in quantifiers) and $|E_Q^-|$ (size of negated edges). (1) We first mined frequent features, including edges and paths of length up to 3 on each of *Pokec* and *YAGO2*. We selected top 5 most frequent features as "seeds", and combined them to form the stratified pattern Q_π of $|V_Q|$ nodes and $|E_Q|$ edges. (2) For frequent pattern edges $e{=}(u, u')$, we assigned a positive quantifier $\sigma(e) \geq p$%, where p% is initialized as 30% unless otherwise specified. This completes the generation of $\Pi(Q)$. (3) We added $|E_Q^-|$ negated edges to $\Pi(Q)$ between randomly selected node pairs (u, u'), to complete the construction of Q. For synthetic graphs, we generated 50 QGPs with labels drawn from $\mathcal{L}$.

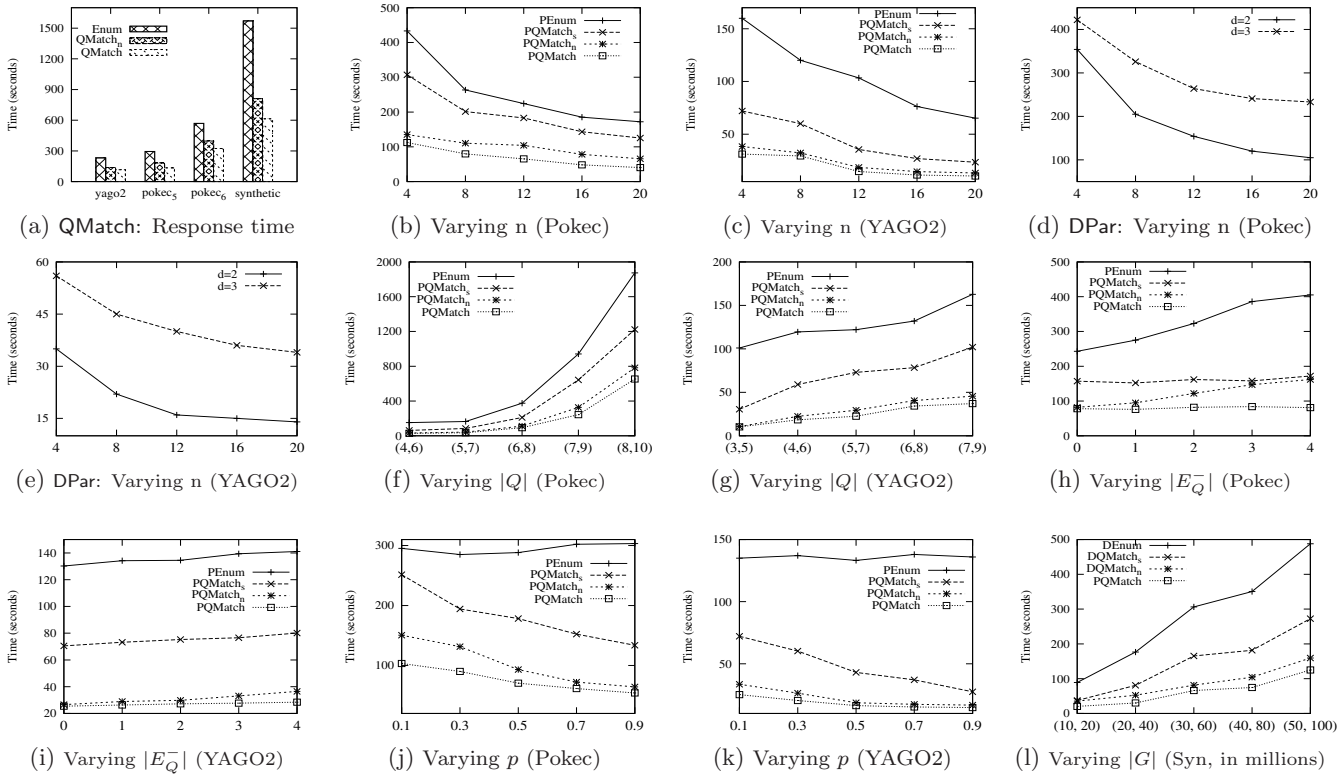
We denote by $|Q| = (|V_Q|, |E_Q|, p_a, |E_Q^-|)$ the size of QGP Q, where p_a is the average of p in all its quantifiers.

Algorithms. We implemented the following, all in Java.

(1) Algorithm QMatch, versus (a) QMatch_n, a revision of QMatch that processes negated edges using DMatch, not the incremental IncQMatch, and (b) Enum, which adopts a state-of-the-art subgraph isomorphism algorithm [35] to enumerate all matches first, and then verify quantifiers. The algorithm in [35] is verified to outperform conventional counterparts, *e.g.,* VF2, by 3 orders of magnitude.

(2) Algorithm PQMatch, versus (a) $\mathsf{PQMatch}_s$, its single-thread counterpart, (b) $\mathsf{PQMatch}_n$, the parallel version of QMatch_n, and (c) PEnum, a parallel version of Enum, which first invokes a parallel subgraph listing algorithm [37] to enumerate all matches, and then verifies quantifiers. We also implemented (d) DPar for d-hop preserving partition.

We deployed the parallel algorithms over n processors for $n \in [4, 20]$. Each processor has 2.6GHz 4vCPU with 16G



(a) QMatch: Response time (b) Varying n (Pokec) (c) Varying n (YAGO2) (d) DPar: Varying n (Pokec)

(e) DPar: Varying n (YAGO2) (f) Varying $|Q|$ (Pokec) (g) Varying $|Q|$ (YAGO2) (h) Varying $|E_Q^-|$ (Pokec)

(i) Varying $|E_Q^-|$ (YAGO2) (j) Varying p (Pokec) (k) Varying p (YAGO2) (l) Varying $|G|$ (Syn, in millions)

Figure 8: Performance evaluation

memory, and 128GB SSD storage. Each experiment was run 5 times and the average is reported here.

Experimental results. We next report our findings.

Exp-1: Performance of QMatch. We first evaluated the performance of QMatch versus QMatch$_n$ and Enum. Fixing $|Q|$=$(5, 7, 30\%, 1)$, *i.e.,* patterns with 5 nodes and 7 edges, p_a = 30% and one negative edge, Figure 8(a) reports the performance of QMatch over two real-world graphs *Pokec* (denote the result as $Pokec_5$) and *YAGO2*, and a larger synthetic graph G_s of 50 million nodes and 100 million edges. We find the following. (1) QMatch outperforms the other algorithms. It is on average 1.2 and 2.0 times faster than QMatch$_n$ and Enum over *YAGO2*, 1.3 and 2.0 times faster over *Pokec*, and 1.3 and 2.6 times faster over G_s, respectively. This verifies that our optimization strategies effectively reduce the verification cost. (2) QMatch works reasonably well over real-world social and knowledge graphs. It takes up to 150 (resp. 116) seconds over *Pokec* (resp. *YAGO2*), comparable to conventional subgraph isomorphism without quantifiers. Moreover, it takes longer time for larger $|Q|$ (*e.g.,* with size $(6, 8, 30\%, 1)$, the result is denoted as $Pokec_6$), as expected.

Exp-2: Scalability of PQMatch. This set of experiments evaluated the scalability of parallel algorithm PQMatch, compared to PQMatch$_n$, PQMatch$_s$, and PEnum. In these experiments, we fixed $|Q| = (6, 8, 30\%, 1)$, $d = 2$ for d-hop preserving partition and $b = 4$ for the number of threads in intra-fragment parallelism, unless stated otherwise.

*Varying n (*PQMatch*).* We varied the number n of processors from 4 to 20. As shown in Fig. 8(b) (resp. Fig. 8(c)) over *Pokec* (resp. *YAGO2*), (1) PQMatch and PQMatch$_s$ scale well with the increase of processors: for PQMatch, the improvement is 2.8 (resp. 3.2) times when n increases from 4 to 20; this verifies Theorem 7; (2) PQMatch is 3.8 (resp. 5.8) times faster than PEnum; and (3) with optimization strategies (incremental evaluation and multi-threads), PQMatch outperforms PQMatch$_n$ and PQMatch$_s$ by 1.5 (resp. 1.1) times and 2.8 (resp. 2.3) times, respectively. (4) PQMatch works reasonably well on large graphs. With 20 processors, it takes 40.3 (resp. 10.2) seconds on *Pokec* (resp. *YAGO2*).

*Varying n (*DPar*).* We also evaluated the scalability of DPar for d-hop preserving partition, with $d = 2$ and $d = 3$. Here DPar incrementally computed the partition when d is changed from 2 to 3 (see Section 5.1). As shown in Figures 8(d) and 8(e), (1) DPar scales well with n: when d=2, the improvement is 3.5 (resp. 2.5) times when n increases from 4 to 20 over *Pokec* (resp. *YAGO2*). (2) The fragments are well balanced: the "skew" (the ratio of the size of the smallest fragment to the largest one) is at least 80% when n=8, for both *Pokec* and *YAGO2*. (3) DPar effectively updates the partition when d is increased.

These justify the parallel scalability of DPar and PQMatch.

Varying $|Q|$. Fixing p_a = 30%, $|E_Q^-| = 1$ and $n = 8$, we varied $(|V_Q|, |E_Q|)$ from (4, 6) to (8, 10) (resp. (3, 5) to (7, 9)) on *Pokec* (resp. *YAGO2*). As shown in Figures 8(f) and 8(g), (1) the larger $|Q|$ is, the longer time is taken by all the algorithms, as expected. (2) PQMatch works well on real-life queries. For queries of size $(5, 7)$ (close to real-world queries), it takes up to 35 (resp. 16.3) seconds over *Pokec* (resp. *YAGO2*). It works better on more sparse *YAGO2*. (3) PQMatch outperforms the other algorithms, which is consistent with the results shown in Figures 8(b) and 8(c).

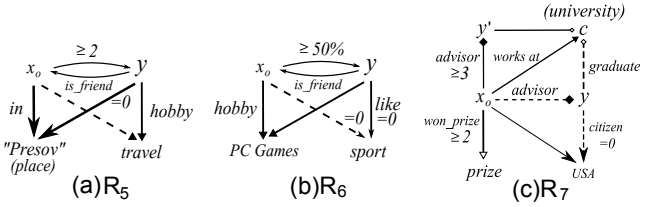


Figure 9: Real-world QGARs

Varying $|E_Q^-|$. We also studied the impact of the number of negated edges. The purpose of this test is to evaluate the effectiveness of incremental matching strategy IncQMatch.

Fixing $n = 8$, $(|V_Q|, |E_Q|) = (6, 8)$ and $p_a = 30\%$, we varied $|E_Q^-|$ from 0 to 4 by selecting $|E_Q^-|$ edges e and "negating" them by setting $\sigma(e) = 0$. As shown in Figures 8(h) and 8(i), (1) PQMatch and PQMatch_s are rather indifferent to the change of $|E_Q^-|$, which incurs small extra cost due to their incremental strategy (IncQMatch). (2) In contrast, PQMatch_n and PEnum are more sensitive to the increment of $|E_Q^-|$. Both algorithms, without IncQMatch, always recompute the matches of pattern Q^{+e} for each negated edge $e \in E_Q^-$, and hence take more time over larger $|E_Q^-|$. The improvement of PQMatch over PQMatch_n and PEnum becomes more significant (from 1.1 to 2 times, and 3.1 to 5 times) with larger $|E_Q^-|$ (from 1 to 4) over *Pokec*. These results verify the effectiveness of IncQMatch.

Varying p_a. Fixing n=8, $|E_Q^-| = 1$ and $(|V_Q|, |E_Q|) = (6, 8)$ (resp. (5, 7)) for *Pokec* (resp. *YAGO2*), we evaluated the impact of aggregates by varying p_a from 10% to 90%. As shown in Figures 8(j) and 8(k), (1) with larger p_a, PQMatch, PQMatch_s and PQMatch_n take less time, since more candidates are pruned in the verification process. (2) In contrast, PEnum is indifferent to the change of p_a, since it always enumerates all the matches regardless of p_a. This verifies the effectiveness of the pruning strategies of PQMatch.

Observe that PQMatch is less sensitive than PQMatch_n to p_a. When p_a is small, the overhead of PQMatch_n incurred by the recomputation of Q^{+e} for negated edges e is larger, since a large number of candidates need to be verified. With larger p_a (more strict quantifiers), the overhead reduces due to the effective pruning of candidates by PQMatch_n. This explains the comparable performance of PQMatch and PQMatch_n when p_a is large (*e.g.,* p_a=0.9).

Varying $|G|$. Fixing $n = 4$, we varied $|G|$ from $(10M, 20M)$ to $(50M, 100M)$ using synthetic social graphs. As shown in Fig. 8(l), (1) PQMatch scales well with $|G|$ and is feasible on large graphs. It takes 125 seconds when $|G| = (50M, 100M)$. (2) PQMatch is 1.5, 2.3 and 4.7 times faster than PQMatch_n, PQMatch_s and PEnum on average, respectively.

Exp-3: Effectiveness of QGAR. We also evaluated the effectiveness of QGARs. We developed a simple QGAR mining algorithm by extending the algorithm of [16] for mining graph pattern association rule (GPARs). GPARs are a special case of QGARs $Q_1(x_o) \Rightarrow Q_2(x_o)$ that have no quantifiers and restrict Q_2 to a single edge. (1) We mined a set of top GPARs using [16] over *Pokec* and *YAGO2*, for confidence threshold $\eta = 0.5$. For each GPAR R, we initialized a QGAR R'. (2) We extended Q_2 in each R' by adding frequent edges whenever possible, and by gradually enlarging p_a for frequent edges by increment 10% (1 for numeric aggregates). We stopped when the confidence of R' got below η. We show three QGARs in Fig. 9, illustrated as follows.

(1) R_5 (*Pokec*) says that if a user has "long-distance" friends, *i.e.,* at least two of her friends do *not* live in the same city "Presov" where she lives, then the chances are that they share the hobby of traveling. We found 50 matches in *Pokec*.

(2) R_6 (*Pokec*; confidence 0.8) demonstrates a negative pattern: for a user x_o, if more than half of his friends share the same hobby "PC Games", and none of them like sports, then it is likely x_o does not like sports. R_6 has support 4000.

(3) R_7 (*YAGO2*; confidence 0.75) states that if a US professor (a) won at least two academic prizes, and (b) graduated at least 4 students, then the chances are that at least one of her/his students is not a US citizen. It discovers scientists such as Marvin Minsky (Turing Award 1969) and Murray Gell-Mann (Nobel Prize Physics 1969) from *YAGO2*. Here Q_2 in R_7 has three (dashed) edges, as opposed to GPARs [16].

These QGARs demonstrate quantified correlation between the entities in social and knowledge graphs, which cannot be captured by conventional association rules and GPARs [16].

Summary. We find the following. Over real-life graphs, (1) quantified matching is feasible: PQMatch (with 20 processors) and QMatch took 40.3s and 342s on *Pokec*, and 10.2s and 116s on *YAGO2*, respectively. (2) Better still, PQMatch and DPar are parallel scalable: their performance is improved by 3 times on average with workers increased from 4 to 20. (3) Our optimization techniques improve the performance of QMatch and PQMatch by 1.27 and 1.3 times on average, and 2.2 and 4.5 times over Enum and PEnum, respectively. (4) QGARs capture behavior patterns that cannot be expressed with conventional graph patterns.

8. CONCLUSION

We have proposed quantified matching, by extending traditional graph patterns with counting quantifiers. We have also studied important issues in connection with quantified matching, from complexity to algorithms to applications. The novelty of this work consists in quantified patterns (QGPs), quantified graph association rules (QGARs), and algorithms with provable guarantees (*e.g.,* optimal incremental matching and parallel scalable matching). Our experimental study has verified the effectiveness of QGPs and the feasibility of quantified matching in real-life graphs.

We will study practical extensions of QGPs to more general graph patterns (*e.g.,* bounded simulation with regular path constraints) and other built-in predicates ($<, \leq, \neq$). Another topic concerns QGAR discovery. It calls for a nontrivial extension of prior pattern mining algorithms (*e.g.,* [16, 17]) to accurately identify quantifiers.

Acknowledgments. Fan and Xu are supported in part by ERC 652976, 973 Program 2014CB340302 and 2012CB316200, NSFC 61133002 and 61421003, EPSRC EP/J015377/1 and EP/M025268/1, NSF III 1302212, Shenzhen Peacock Program 1105100030834361, Guangdong Innovative Research Team Program 2011D005, Shenzhen Science and Technology Fund JCYJ20150529164656096 and Guangdong Applied R&D Program 2015B010131006.

9. REFERENCES

[1] Nielsen global online consumer survey. *http://www.nielsen.com/content/dam/corporate/us/en/newswire/uploads/2009/07/pr_global-study_07709.pdf.*

[2] Pokec social network. *http://snap.stanford.edu/data/soc-pokec.html.*

[3] R. Agrawal, T. Imieliński, and A. Swami. Mining association rules between sets of items in large databases. *SIGMOD Record*, 22(2):207–216, 1993.
[4] W. Amaldoss and S. Jain. Research note-trading up: A strategic analysis of reference group effects. *Marketing science*, 27(5):932–942, 2008.
[5] S. Amer-Yahia, L. V. S. Lakshmanan, and C. Yu. Socialscope: Enabling information discovery on social content sites. In *CIDR*, 2009.
[6] K. Andreev and H. Racke. Balanced graph partitioning. *Theory of Computing Systems*, 39(6):929–939, 2006.
[7] D. A. Bader and K. Madduri. Gtgraph: A synthetic graph generator suite.
[8] R. Bapna and A. Umyarov. Do your online friends make you pay? A randomized field experiment in an online music social network. *NBER working paper*, 2012.
[9] M. Bendersky, D. Metzler, and W. Croft. Learning concept importance using a weighted dependence model. In *WSDM*, 2010.
[10] H. Blau, N. Immerman, and D. Jensen. A visual language for querying and updating graphs. *University of Massachusetts Amherst Computer Science Technical Report*, 37:2002, 2002.
[11] F. Bourse, M. Lelarge, and M. Vojnovic. Balanced graph edge partition. In *SIGKDD*, pages 1456–1465, 2014.
[12] P. Burkhardt and C. Waring. An NSA big graph experiment. Technical Report NSA-RD-2013-056002v1, U.S. National Security Agency, 2013.
[13] C. Chekuri and S. Khanna. A PTAS for the multiple knapsack problem. In *SODA*, pages 213–222, 2000.
[14] X. Dong et al. Knowledge vault: A web-scale approach to probabilistic knowledge fusion. In *KDD*, 2014.
[15] W. Fan, X. Wang, and Y. Wu. Incremental graph pattern matching. *TODS*, 38(3):18, 2013.
[16] W. Fan, X. Wang, Y. Wu, and J. Xu. Association rules with graph patterns. *PVLDB*, 2015.
[17] L. A. Galárraga, C. Teflioudi, K. Hose, and F. Suchanek. AMIE: association rule mining under incomplete evidence in ontological knowledge bases. In *WWW*, 2013.
[18] M. A. Gallego, J. D. Fernández, M. A. Martínez-Prieto, and P. de la Fuente. An empirical study of real-world SPARQL queries. In *USEWOD workshop*, 2011.
[19] A. Goyal, F. Bonchi, and L. V. Lakshmanan. Discovering leaders from community actions. In *CIKM*, 2008.
[20] I. Grujic, S. Bogdanovic-Dinic, and L. Stoimenov. Collecting and analyzing data from e-government facebook pages. In *ICT Innovations*, 2014.
[21] M. R. Henzinger, T. Henzinger, and P. Kopke. Computing simulations on finite and infinite graphs. In *FOCS*, 1995.
[22] J. Huang, D. J. Abadi, and K. Ren. Scalable SPARQL querying of large RDF graphs. *PVLDB*, 4(11), 2011.
[23] G. Karypis. Metis and parmetis. In *Encyclopedia of Parallel Computing*, pages 1117–1124. 2011.
[24] R. E. Korf. Minimizing disk I/O in two-bit breadth-first search. In *AAAI*, pages 317–324, 2008.
[25] F. Korn, A. Labrinidis, Y. Kotidis, and C. Faloutsos. Ratio rules: A new paradigm for fast, quantifiable data mining. In *VLDB*, 1998.
[26] C. P. Kruskal, L. Rudolph, and M. Snir. A complexity theory of efficient parallel algorithms. *TCS*, 71(1), 1990.
[27] J. Lee, W.-S. Han, R. Kasperovics, and J.-H. Lee. An in-depth comparison of subgraph isomorphism algorithms in graph databases. In *PVLDB*, volume 6, 2012.
[28] C. Ley, B. Linse, and O. Poppe. SPARQLog: SPARQL with rules and quantification. *Semantic Web Information Management: A Model-Based Perspective*, page 341, 2010.
[29] L. Libkin. *Elements of Finite Model Theory*. Springer, 2004.
[30] W. Lin, S. A. Alvarez, and C. Ruiz. Collaborative recommendation via adaptive association rule mining. *Data Mining and Knowledge Discovery*, 6:83–105, 2000.
[31] V. Liptchinsky, B. Satzger, R. Zabolotnyi, and S. Dustdar. Expressive languages for selecting groups from graph-structured data. In *WWW*, 2013.
[32] M. Martın, C. Gutierrez, and P. Wood. SNQL: A social networks query and transformation language. In *AMW*, 2011.
[33] C. H. Papadimitriou. *Computational Complexity*. Addison-Wesley, 1994.
[34] G. Ramalingam and T. Reps. On the computational complexity of dynamic graph problems. *TCS*, 158(1-2), 1996.
[35] X. Ren and J. Wang. Exploiting vertex relationships in speeding up subgraph isomorphism over large graphs. *PVLDB*, 8(5):617–628, 2015.
[36] C. Schmitz, A. Hotho, R. Jäschke, and G. Stumme. Mining association rules in folksonomies. In *Data Science and Classification*, pages 261–270. 2006.
[37] Y. Shao, B. Cui, L. Chen, L. Ma, J. Yao, and N. Xu. Parallel subgraph listing in a large-scale graph. In *SIGMOD*, 2014.
[38] R. Srikant and R. Agrawal. Mining quantitative association rules in large relational tables. In *SIGMOD*, 1996.
[39] F. M. Suchanek, G. Kasneci, and G. Weikum. Yago: a core of semantic knowledge. In *WWW*, pages 697–706, 2007.
[40] X. Wu, C. Zhang, and S. Zhang. Efficient mining of both positive and negative association rules. *TOIS*, 22(3), 2004.

Appendix A: Proofs

Proof of Theorem 1

It is easy to see that Theorem 1 follows from Lemmas 2, 3 and 4. Below we verify these lemmas.

Lemma 2: *Lower bounds*. We first prove the lower bounds of quantified matching, for positive and negative QGPs.

(1) NP-hardness. The quantified matching for positive QGPs with numeric aggregates is already NP-hard. This can be verified by constructing a polynomial time reduction from subgraph isomorphism (**subISO**), where a QGP Q is constructed by adding quantifiers $\delta(e) \geq 1$ for all the edges e in a graph pattern Q' in an instance of **subISO**. As **subISO** is NP-hard (cf. [33]), so is quantified matching.

(2) DP-hardness. We next prove that quantified matching for (possibly) negative QGPs is DP-hard by reduction from Exact-Clique, which is DP-complete (cf. [33]). Given a graph G and a natural number k, Exact-Clique is to determine whether the largest clique of G has size exactly k. Given an instance of Exact-Clique, We construct a QGP $Q(x_o)$ that consists of a k-clique Q_1, a $k+1$-clique Q_2 such that x_o has a unique match v_o in G', there is an edge e from x_o to each node in q_1 with $\sigma(e) \geq 1$, and to a single node v' in Q_2 via a negated edge. The transformation is obviously in PTIME. Moreover, the largest clique of G has size k if and only if $v_o \in Q(x_o, G')$. Therefore, the transformation above is a reduction. As Exact-Clique is DP-complete, quantified matching with (possibly) negative QGPs is DP-hard.

Lemma 3: *Upper bounds*. We next prove the upper bounds for quantified matching. We first consider positive QGPs Q with numeric aggregates $\sigma(e) \odot p$ only (Lemma 3). We then extend the result to ratio aggregates $p\%$ (Lemma 4).

(1) Given a QGP $Q(x_o)$, we construct a traditional graph pattern $Q_e(x_o)$ without quantifiers, by (a) stripping off all quantifiers from Q, and (b) for each edge $e(u, u')$ associated with $\sigma(e) \odot p$, if $p > 1$, we make p copies of u' in Q_e as children of u, along with copies of edges from u' and so on. Then one can easily verify the following: (a) for any

graph G, $Q(x_o, G) = Q_e(x_o, G)$, and (b) the time for constructing Q_e and hence $|Q_e|$ are both a polynomial in $|Q|$; this is because on each simple path in Q, there are at most k non-existential quantifiers, for a predefined constant k. These make a PTIME reduction from quantified matching with positive numeric aggregates to conventional subgraph isomorphism. Since the latter is in NP, so is the former.

(2) We next prove that the quantified matching for (possibly) negative QGPs is in DP. Following [33], it suffices to construct two languages L_1 and L_2, such that a node v_x is in $Q(x_o, G)$ if and only if v_x is in $L_1 \cap L_2$.

We consider two languages below:

- L_1, the set $\Pi(Q)(x_o, G)$, and
- L_2, the set of "yes" instances for a node v_x that is not a match of x_o for $\bigcup_{e \in E_Q^-} (\Pi(Q^+)(x_o, G)$.

One can verify that (1) $L_1 \in$ NP, (2) $L_2 \in$ coNP, and (3) a node v_x is in $Q(x, G)$ if and only if v_x is in $L_1 \cap L_2$, by the definition of QGPs. Thus quantified matching is in DP.

Lemma 4: Ratio aggregates. Given a QGP Q that contains ratio aggregates $\sigma(e) \odot p\%$, we construct a QGP Q_d and graph G_d in PTIME such that Q_d consists of numeric aggregates only, and $Q_d(x_o, G_d) = Q(x_o, G)$. To simplify the discussion, we consider *w.l.o.g.* positive Q. For negated edges e, by the definition of $Q(x_o, G)$, e is positified in Q^{+e}. Hence, it suffices to consider positive edges.

(a) We transform G to a graph G_d as follows. For each node v with g child in G, we add $(1-p\%)(d-g)$ dummy children with a label that does not match any pattern node in Q, and $p\%(d-g)$ dummy children that complete a dummy subgraph G_Q at v that is isomorphic to Q_π of Q.

(b) We transform Q to Q_d such that for each edge e with quantifier $\sigma(e) \odot p\%$, we replace $p\%$ with a constant $p\% * d$.

One may verify that a node $v_x \in \mathcal{Q}(x_o, G)$ if and only if its G_d-counterpart $v_d \in Q_d(x_d, G_d)$. Moreover, the transformation is obviously in PTIME. Since quantified matching for all numeric quantified Q_d is in NP by Lemma 3, so is its counterpart for QGPs Q with ratio quantifiers. □

This completes the proof of Theorem 1.

Proof of Lemma 5

To show the correctness of DMatch, first observe that DMatch always terminates. Indeed, DMatch follows the verification process of conventional subgraph isomorphism algorithm. The process, in the worst case, enumerates all possible isomorphism mappings from the stratified pattern Q_π to G, which are finitely many. Hence DMatch terminates.

We next show that DMatch correctly verifies whether a candidate v_x is a match of x_o in $\Pi(Q)$ via an isomorphism $h_0 \in \Pi(Q)(G)$. It suffices to show that (1) h_0 is a match in $Q_\pi(G)$, and (2) for each u in $\Pi(Q)$ and each edge $e{=}(u, u')$, $|M_e(h_0(x_o), h_0(u), Q)| \odot p$ for $f(e) = \sigma(e) \odot p$.

(1) When DMatch terminates, for each $u \in \Pi(Q)$ and every candidate v in $C(u)$ with $X(u, v){=}$true, $v = h(u)$ for some $h \in Q_\pi(G)$, guaranteed by the correctness of Match.

(2) For each edge (u, u') in $\Pi(Q)$ and a node v with $X(u, v){=}$true, DMatch correctly verifies the quantifiers by checking the updated local counter of v that keeps track of the current $|M_e(h_0(x_o), h_0(u), Q)|$. In addition, DMatch *waits* until either v is determined not a valid match due to that the upper bound fails the quantifier (by the local pruning rule, Appendix B), or the lower bound satisfies the quantifier (in the verification). Hence, v_x is a match if and only if $v_x \in \Pi(Q)(x_o, G)$ when DMatch terminates.

For the space complexity, it takes $O(|V|)$ space to store the auxiliary structures for the nodes in G. During the search, DMatch keeps, at each level of the search, at most p_m best matches to be verified, where p_m is the largest constant in quantifiers. Since there are in total $|\Pi(Q)| \leq |Q|$ levels of search, it takes in total $O(p_m|Q| + |V|)$ space. □

Proof of Proposition 6

We prove Proposition 6 by giving the correctness and complexity analysis of IncQMatch below.

Correctness. Given a QGP $\Pi(Q^{+e})$, algorithm IncQMatch correctly computes $\Pi(Q^{+e})(x_o, G)$ by processing ΔE one edge at a time. At any time, the newly formed pattern Q' contains $\Pi(Q)$ as a subgraph. It is easy to verify that $\Pi(Q)(u, G) \subseteq \Pi(Q^{+e})(u, G)$, for any $u \in \Pi(Q)$. Hence, IncQMatch only needs to determine the nodes to be removed from the cached matches/candidates from QMatch.

We next show that IncQMatch removes a node v from $C(u)$ if and only if it is not a match in $\Pi(Q^{+e})(u, G)$, for any node u in $\Pi(Q^{+e})$. (1) If v is not a match, then either v is not in an isomorphism mapping, or v fails the quantifier of at least one edge (u, u'). IncQMatch captures both cases by the isomorphism checking and quantifier verification. Hence it guarantees to remove all the v that are not match. (2) Assume by contradiction that IncQMatch removes a node v that is a match. Then either v is not a match via isomorphism, or v fails the counter for an edge (u, u'). Both contradict the assumption that v is a match. Hence, IncQMatch only removes the nodes that are not matches in $\Pi(Q^{+e})(u, G)$.

Complexity. During the process, IncQMatch visits and verifies the following sets of nodes: (1) $C(u)$, including the cached matches and candidates of u if u is not in $\Pi(Q)$, where u is in edge $e = (u, u')$ or $e{=}(u', u)$, for each $e \in \Delta E$; and (2) those nodes reachable from (or can be reached by) those nodes in (1) via a sequence of cached matches/candidates including those nodes in (1). The number of verification hence is bounded by the size of the set combining (1) and (2), which is in total at most |AFF|. □

Proof of Theorem 7

We prove Theorem 7 by providing the correctness and complexity analysis below for algorithm PQMatch.

Correctness. Given graph G distributed over n processors by a d-hop preserving partition P_d, PQMatch computes $Q(x_o, G)$ as $\bigcup Q(x_o, F_i)$ $(i \in [1, n])$, for any QGP $Q(x_o)$ with radius bounded by d. It suffices to show Lemma 9(1).

Lemma 9(1). Observe the following. (1) For any match $v_x \in Q(x_o, F_i)$, QMatch only needs to visit $N_d(v_x)$ to verify whether v_x is a match. (2) For every candidate $v_x \in C(x_o)$, there exists a fragment F_i, such that $N_d(v_x) \subseteq F_i$ (including v_x) (by d-hop preservation). Hence, any match of x_o must be from at least one match set $Q(x_o, F_i)$ evaluated at fragment F_i. (3) For every match $v_x \in Q(x_o, F_i)$ locally computed at F_i, v_x is a match of x_o guaranteed by the correctness of QMatch. Hence PQMatch correctly computes $Q(x_o, G)$ as $\bigcup Q(x_o, F_i)$ $(i \in [1, n])$ over a d-hop preserving partition.

Complexity. Algorithm PQMatch consists of three steps: (1) the distribution of Q and construction of d-hop preserving partition, (2) the parallel evaluation, and (3) assembling of partial matches. The time for step (2) and (3) are in $O(\frac{t(Q,G)}{n})$ and $O(n)$, respectively, *i.e.,* are parallel scalable.

Hence it suffices to focus on the parallel scalability of step (1), by proving Lemma 8 and Lemma 9(2).

Lemma 8 (Parallel scalability of DPar*).* We first show that procedure DPar is parallel scalable. We will show the approximation ratio of DPar separately in the next proof.

Observe the following. (1) For each worker S_i managing a fragment F_i in the base partition, the border nodes $F_i.O$ can be computed via a linear scan of F_i. Hence the overall time is in $O(|C_d * \frac{|G|}{n}|)$ by the condition of Lemma 8. (2) Given $V'=\bigcup F_i.O$, DPar applies the $(1+\epsilon)$ approximation algorithm of [13], which computes an assignment in time $O(|V'|^{\frac{1}{\epsilon}})$. For ϵ small enough for a good approximation, *e.g.,* ϵ=1, the time cost is $O(|V'|)$. Since $|V'|=\sum F_i.O \leq \sum N_d(v)$ ($v \in V'$), and $\sum N_d(v)$ is bounded by $O(C_d * \frac{|G|}{n})$, the process takes time in $O(C_d * \frac{|G|}{n}) = O(\frac{|G|}{n})$. (3) For each border node $v \in F_i.O$, each S_i fetches $N_d(v)$ from G in parallel. In the worst case, each worker takes in total $O(\sum |N_d(v)|)$ time for all the border nodes $v \in F_i.O$. As the fetch process is bounded by $O(C_d * \frac{|G|}{n})$ at each fragment, the overall parallel partition time is bounded by $O(C_d * \frac{|G|}{n}) = O(\frac{|G|}{n})$. Hence, DPar is parallel scalable.

Lemma 9(2) (Parallel scalability of mQMatch). Procedure mQMatch is conducted locally in parallel at each worker. From the correctness of Lemma 9(1), each worker only performs local matching without the need to communicate with others once DPar terminates. Hence, the overall time complexity is $O(\frac{t(Q,G)}{n})$. The time cost for merging the answers is in $O(n)$ time. Putting these together, PQMatch is in $O(\frac{t(Q,G)}{n}) + O(n)$ time. Lemma 9(2) thus follows, and so does the parallel scalability of PQMatch. □

Lemma 8 (Approximation). The d-hop preserving partition problem at the coordinator S_c is to find an assignment for each $N_d(v)$ to a worker S_i, such that (a) $\sum |N_d(v_i)| \leq c * \frac{|G|}{n}$ for all $N_d(v_i)$ assigned to S_i, and (b) $|V_c|$ is maximized, where V_c refers the nodes covered in V'. We show that the problem is $1+\epsilon$-approximable, by constructing an approximation preserving reduction (APR) [13] to the multiple knapsack problem (MKP) as follows.

- **function** f: (a) for each $v \in V'$, construct an item u_i with value 1 and weight $|N_d(v)|$; and (b) for each worker S_i, construct a bin B_i with capacity $c * \frac{|G|}{n} - |F_i|$.
- **function** g: for each item u_i packed to a bin B_i in $s(I_2)$, g maps u_i to v_i, and B_i to S_i.

We next show that the transformation above is an APR. Indeed, (1) f is in PTIME, and (2) for a feasible solution $s(I_2)$, $g(s(I_1))$ is also a feasible solution, since the packing does not exceed the capacity constraints in $s(I_2)$ if and only if the assignment $g(s(I_1))$ does not exceed the capacity of each fragment. (3) Assume $s(I_2) \geq \frac{s^*(I_2)}{1+\epsilon}$ (α=1). One can verify that $|s^*(I_1)| = |V_c|$ (the size of covered nodes) $= |g(s^*(I_2)|$ (the size of packed items), $|s(I_1)| = |g(s(I_2)|$. Hence $g(s(I_2)) = s(I_1) \geq \frac{s^*(I_2)}{1+\epsilon} = \frac{g(s^*(I_2))}{1+\epsilon}$. Thus, the transformation is an APR. As a result, from [13] it follows that d-hop preserving partition is $1+\epsilon$ approximable.

Remarks. We use a balanced bound for all fragments. This guarantees the correctness of the reduction to MKP. The choice of MKP is to get a balanced fragment bound and at the same time, to minimize synchronization cost. Our experimental study shows that this leads to quite balanced fragments (Exp-2), with minimum communication cost.

Proof of Lemma 10

Assume that $R'(x_o)$: $Q_1'(x_o) \Rightarrow Q_2'(x_o)$ is extended from $R(x_o)$: $Q_1(x_o) \Rightarrow Q_2(x_o)$. It suffices to prove that $R'(x_o, G) \subseteq R(x_o, G)$ for any G. Consider two cases below.

(1) If Q_1' is extended by adding new edge (either positive or negative) to Q_1, then $Q_1'(x_o, G) \subseteq Q_1(x_o, G)$ (see Section 4.2). Hence, $R'(x_o, G) = Q_1'(x_o, G) \cap Q_2(x_o, G) \subseteq Q_1(x_o, G) \cap Q_2(x_o, G) = R(x_o, G)$.

(2) Assume that Q_1' is obtained by increasing p in a positive quantifier of an edge $e=(u, u')$ in Q_1. Assume that there exists a node $v \in R'(x_o, G)$ that is not a match in $R(x_o, G)$. Then there must exist a positive edge (x_o, u) with quantifier $\sigma(e) \odot p$ for which v does not have enough matches in its children. Nevertheless, for any positive quantifier $\sigma(e) \odot p$ in R and its counterpart $\sigma(e) \odot p'$ in R', $p' \geq p$. As v is a match of $R'(x_o, G)$, this contradicts the assumption that v is in $R(x_o, G)$. Hence $R'(x_o, G) \subseteq R(x_o, G)$.

The same argument applies to the case when Q_2' is revised from Q_2. Hence Lemma 10 follows. □

Proof of Corollary 11

As a constructive proof, we outline two algorithms for computing $R(x_o, \eta, G)$ with the desired complexity as follows.

Sequential quantified entity matching. Given a QGAR R, confidence threshold η and G, the first algorithm, denoted as garMatch, (1) invokes QMatch to compute $Q_1(x_o, G)$ and $Q_2(x_o, G)$, respectively; (2) computes $R(x_o, G) = Q_1(x_o, G) \cap Q_2(x_o, G)$; and (3) verifies whether $\mathsf{conf}(R) = \frac{|R(x_o,G)|}{|Q_1(x_o,G) \cap X_o|} \geq \eta$. If so, it returns $R(x_o, G)$.

The correctness and complexity of garMatch follow from their QMatch counterparts (Lemmas 5 and 6).

Parallel quantified entity matching. We introduce a parallel algorithm, denoted as dgarMatch, for parallel quantified entity matching. It follows the generic steps of PQMatch. The only difference is as follows: (a) each worker evaluates two patterns Q_1 and Q_2 in parallel, and (b) the coordinator S_c assembles the results to evaluate the confidence of R.

Algorithm dgarMatch starts with a set of base partitions. (1) It constructs a d-hop preserving partition, where d is a predefined upper bound of the largest radius Q_1 and Q_2 in R. (2) Each worker then computes local match $Q_1(x_o, F_i)$ and $Q_2(x_o, F_i)$ in parallel. It also computes the local set X_{oi}. (3) Each worker returns the local matches to the coordinator S_c. Then dgarMatch computes $R(x_o, G)$ as $(\bigcup Q_1(x_o, F_i)) \setminus (\bigcup Q_2(x_o, F_i))$, and computes the confidence $\mathsf{conf}(R, G)$ as $\frac{|R(x_o,G)|}{|\bigcup Q_1(x_o,F_i) \cap \bigcup X_{oi}|}$. It next verifies whether $\mathsf{conf}(R, G) \geq \eta$ and if so, returns $R(x_o, G)$. Otherwise, it returns $\emptyset$.

The correctness and complexity of dgarMatch follow from their PQMatch counterparts. □

Appendix B: Optimization Techniques Selection and Pruning Rules (Section 4)

We show that the number of verification in algorithm QMatch can be further reduced, by applying the following selection and pruning rules.

(1) [**pruning rules**]. The pruning rules are applied *each time* when DMatch backtracks from a candidate $v' \in S_P(u')$ to a match $v \in S_P(u)$ for an edge $e=(u,u')$.

(a) *local pruning rule*: If the dynamically maintained upper bound $U(v_x,e)$ of each $v_x \in C(x_o)$ fails the quantifier of e (because of a verified non-match v' as its child), then removes v from $C(u)$ and $S_P(u)$.

(b) *global pruning rule*: monitors the size of current candidates $C(u')$, excluding those nodes verified to be non-matches (*i.e.,* $X(u',u)=\mathsf{false}$).

The lemma below justifies these pruning rules.

Lemma 12: *Node x_o in $\Pi(Q)$ has a match only if for every node u' in $\Pi(Q)$, $|C(u')| \geq p_m$, where p_m=max$\{p'\}$ for p' ranging over p in $\sigma(u,u') \geq p$ for all u''s parent u.* □

If $C(u') < p_m$ for some pattern node u', QMatch terminates and returns the current $\Pi(Q)(x_o,G)$.

(2) [**selection rule**]. The selection rule is applied *each time* when DMatch extends the search to a next level, at a match $v \in S_P(u)$ for edge $e=(u,u')$. Denote as $P(u')$ the parent set of u' in G, the *potential* of a match $v' \in C(u')$ is

$$(1+\frac{|P(v')\cap C(u)|}{|C(u)|}) * \Sigma_{\forall e=(u',u'')}\frac{U(v',e)}{p_e}$$

The rule picks candidates with highest potential scores.

Example 10: Consider Q_4 with p=3 (Fig. 2) and a revised G_2 (Fig. 2) by changing "UK" to "US". QMatch terminates early without verifying all the nodes, in contrast to Match. As the potential of the candidates x_4, x_5 and x_6 for x_o are all 1, QMatch starts with *e.g.,* x_5. (1) Following edge $e=(x_o,z)$, the potentials of candidates v_8, v_6, v_7 and v_9 are $2*(1+\frac{3}{3})$=4, 4, $\frac{5}{3}$, and $\frac{4}{3}$, respectively. Hence QMatch selects v_6, v_8 and v_7 as top 3 candidates, starting with v_6. (2) When backtracking to v_6, QMatch finds v_6 not a match; so it reduces the upper bound of x_4-x_6 for edge e by 1. QMatch next backtracks to x_5. By the local pruning rule, it finds $U(x_5,e)$ smaller than 3; thus v_5 is not a match. (3) Applying the global pruning rule, the size $C(z)$ ($\{v_7,v_9\}$) is already smaller than 3. Hence QMatch returns $\emptyset$ without further checking. □

Proof of Lemma 12

Proof: We show Lemma 12 by contradiction. Assume that a node $v_x \in C(x_o)$ is a match of x_o, and there exists a node u' in $\Pi(Q)$ that $|C(u')| < p_m$, where p_m is the largest constant in the positive quantifier from edges (u,u'). (1) Since v_x is a match of x_o, there exists a node $v' \in C(u')$ that is a match for u'. As v' is a valid match for u', it must have a parent v that matches u in an isomorphic mapping, and satisfies the quantifier of edge $e_i=(u_i,u')$, for every parent u_i of u'. (2) This indicates that v' must have at least p_i children that match u' with size at least p_i, for e_i with quantifier $\sigma(e_i) \odot p_i$, *i.e.,* $|C(u')| \geq p_i$. (3) Since (2) holds for every edge (u_i,u'), $|C(u')| \geq p_m$. This contradicts the assumption that $|C(u')| < p_m$. Hence Lemma 12 follows. □

Optimization For QMatch (Section 4)

We leverage graph simulation [21] to further reduce verification. A node v in G simulates a query node u in Q, *i.e.,* (u,v) is in a simulation relation R if v and u have the same label, and for each u's child u' connected via edge e, v has a child v' connected via edge e' having the same label of e such that $(u',v') \in R$. It is known that R can be computed in quadratic time [21]. One can verify the following.

Lemma 13: *For any edge $e=(u,u')$ in $\Pi(Q)(x_o)$ and its quantifier $\sigma(e) \odot p$, v is a match of u in Q when x_o is mapped to v_x only if (1) v_x simulates x_o, (2) v simulates u, and (3) $|R(v_x,v,G)|$ (resp. $\frac{|R(v_x,v,F_i)|}{|M_e(v)|}$) $\odot$ p (resp. p%).* □

Here $R(v_x,v,G)$ is the set of children of v simulating u'. Indeed, one may verify that $|R(v_x,v,G)|$ is an upper bound of $|M_e(h_0(x_o),h_0(u),Q)|$ for positive edges e. The result above suggests that we can use simulation as preprocessing, and remove invalid candidates early by following Lemma 13.

Proof of Lemma 13

One may verify that v is a match for u via isomorphism only if v can simulate u. Indeed, If a node v cannot simulate u, then either $v \notin C(u)$, or no descendant u' of u s in $C(u')$. Both indicates that v cannot match u via an isomorphism. Assume that v is a match of u when v_x is matched with x_o. (1) If v_x (resp. v) does not simulate x_o (resp. u), then v_x (resp. v) cannot match x_o (resp. u) via isomorphism. (2) If $|R(v_x,v,F_i)|$ fails the quantifier e, then the number of possible matches in v's children fails the quantifier. Both contradict the assumption. Lemma 13 thus follows. □

Multi-thread Quantified Matching (Section 5)

Given $Q(x_o)$ and fragment F_i, mQMatch follows QMatch to verify the candidates in $C(x_o)$. It dynamically spawns p threads to simultaneously verify the top p selected candidates v for each pattern edge $e=(u,u')$, one for each candidate. For a candidate $v \in C(u)$, the thread i for v spawns a thread for each of the top p candidates $v' \in C(u')$ to be verified. As multi-threading can be expensive when d is large, mQMatch utilizes *thread blocking* to avoid the overhead: (1) it only spawns at most b threads, and "sequentializes" the rest $p-b$ verification with a single thread, and (2) it only spawns threads for the first two search levels.

Appendix C: Confidence of QGARs (Section 6)

One might be tempted to define the confidence of $R(x_o)$ as $\frac{|R(x_o,G)|}{|Q_1(x_o,G)|}$, following traditional association rules [38]. However, this does not work well in incomplete graphs.

Example 11: For QGAR R_1, consider two matches v_1 and v_2 in $Q_1(x_o,G)$, where user v_1 has *no* edge labeled buy. Since G is usually incomplete, it is an overkill to assume that v_1 is a negative example as a potential customer of books, since some of its buy edges may possibly be missing from G. □

To accommodate incomplete graphs, we follow the local close world assumption (LCWA) [14], which assumes that G is locally complete, *i.e.,* either G includes the complete neighbors of a node for any existing edge type, or it knows nothing about the neighbors. We define conf(R,G), the confidence of $R(x_o)$ in G under LCWA, as $\frac{|R(x_o,G)|}{|Q_1(x_o,G)\cap X_o|}$.

Continuing with Example 11, user v_2 is retained in X_o but v_1 is excluded due to missing buy edges. Hence, v_1 is no longer considered to be a negative match under LCWA.